Mathematical Knowledge for Teaching Numbers and Operations: Effects of an Online Teacher Professional Development Program

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Teacher professional development (TPD) programs should address individual knowledge gaps and diverse prior knowledge among participants. E-learning formats have emerged as promising solutions, offering flexibility and accessibility tailored to teachers' needs. While these programs enhance access, ensuring quality experiences and outcomes remains challenging. The study reported in this article analyses the effectiveness of a course within an online teacher professional development program designed to enhance mathematical knowledge for teaching numbers and operations to primary school teachers in Chile. The program aims to develop teachers' mathematical knowledge for teaching (MKT) through contextualised learning activities. A quasi-experimental design was employed to assess the program's effectiveness in improving the MKT of teachers with different specialisation backgrounds. The performance results of the MKT of 99 teachers, both with and without specialisation in mathematics, were analysed before, during, and after participation in the program. The results reveal significant gains in MKT across all participants, regardless of initial MKT levels or specialisation backgrounds, underscoring the inclusivity and efficacy of the systematically designed course. These findings highlight the potential of such structured professional development initiatives to effectively enhance teachers' pedagogical capabilities across diverse backgrounds.

Keywords: mathematical knowledge for teaching • mathematics teachers • online learning • teacher professional development

Introduction

Mathematics education faces challenges in a constantly evolving world, with changing educational needs and the adoption of new teaching paradigms (Darling-Hammond, 2020; Desimone, 2009). As teachers must respond to these evolving needs, teacher professional development (TPD) programs in mathematics education have become an essential component to support continuous learning throughout teachers' professional trajectories (Fransson et al., 2009). Ensuring the inclusivity and effectiveness of these programs, however, remains a central challenge for educational systems, particularly in accommodating the diverse backgrounds and knowledge bases of in-service teachers.

Teacher shortages, especially after the COVID-19 pandemic, have led to the hiring of out-of-field teachers in classrooms where the school system has teachers with and without adequate preparation to teach (e.g., Goos et al., 2020; Wiggan et al., 2021). This trend has been observed in countries such as the

United States and Ireland; however, national reports also indicate that in Chile, the shortage of specialised teachers has been particularly pronounced in public schools, particularly in some regions (Nancuante et al., 2024). The significance of this diversity in teacher knowledge and professionalism directly impacts teaching effectiveness (Ingersoll & Perda, 2008). The diverse knowledge and background profiles of teachers can also be attributed to the quality of pre-service teacher education. For instance, in Chile during the first decade of the 2000s, the lack of regulation of higher education led to the proliferation of low-cost, broad niche areas, such as teacher education, which developed alongside a decrease in the selectivity of applicants (Ruffinelli, 2013). Moreover, most primary school teacher education programs had insufficient mathematics courses to address important topics in the school curriculum (Varas et al., 2008). Although substantial efforts have been made to enhance teacher education over the past 20 years, challenges remain.

Chilean National Diagnostic Test for Preservice Teacher Education (CPEIP, 2022), administered to pre-service elementary school teachers during their last year of training, assesses disciplinary and pedagogical knowledge in mathematics, language, natural sciences, and social sciences. The mathematics section includes questions addressing both content knowledge (e.g., number sense, proportional reasoning, geometry) and pedagogical content knowledge, such as interpreting students' mathematical thinking. The test is mandatory but non-punitive; its purpose is diagnostic, providing universities with feedback to improve their programs rather than to certify or exclude individual students. The achievement on the mathematics test has the widest variation among the four disciplines assessed. Furthermore, the performance is at its lowest level, with an achievement rate of 44.9%, indicating that more than half of the students scored below this level.

Given the low achievement in mathematics of pre-service elementary school teachers and the impact of mathematics knowledge for teaching (MKT) on teachers' effectiveness in teaching from the outset of their careers (Hill et al., 2005; Campbell & Lee, 2017; König et al., 2020), Chile has a pronounced need to provide support for enhancing teachers' MKT. This underscores the need for TPD programs that can address individual knowledge gaps, promoting learning in teachers with varying prior knowledge (Campbell & Lee, 2017; Desimone & Garet, 2015). Hence, considering the diversity of initial MKT among potential participants is essential when designing a mathematics-focused professional development (PD) program, especially if it is widely available.

Among the diverse models of TPD programs, those delivered in an e-learning format have emerged as a promising, adaptable solution tailored to participants' needs (Copur-Gencturk et al., 2024; Powell & Bodur, 2019). These models leverage digital technologies to provide flexible, accessible learning and offer valuable opportunities for teachers to deepen their disciplinary knowledge, develop new pedagogical skills, and adapt to changes in the educational environment (Czerniewicz et al., 2019; Powell & Bodur, 2019). Moreover, these programs have gained significance as a means of providing quality training to teachers who may not otherwise have access, due to the country's demographic characteristics or a lack of local training opportunities (Martínez et al., 2021). Access to these TPD programs does not guarantee quality experiences or outcomes and may create a false sense of effectiveness if technology is used merely as a delivery tool without effective design or implementation principles (Powell & Bodur, 2019). Bragg et al. (2021) conducted a systematic literature review to uncover the successful design and delivery of online TPD. They reported that program characteristics that lead to teachers' improved content and pedagogical content knowledge include being content-focused, considerate of differences among individual learners (including differentiated activities and applications), and incorporation of strong engagement practices (such as synchronous meetings and discussion boards). Nonetheless, there are few methodologically sound studies regarding the effectiveness of online TPD (Bragg et al., 2021). Thus, there is a need for empirical evidence to support the identification of the design features of effective online TPD.

The study reported in this article evaluated the efficacy of a content-focused course in an online TPD program, examining its influence on MKT enhancement among participants with diverse specialisation backgrounds and varying levels of initial knowledge. The course, whose instructional design and delivery strategy were discussed, was designed to teach MKT associated with the multiplication and division of whole numbers. The study addresses the following question:

What is the impact of a three-month online teacher professional development course on the mean score of teachers' performances in mathematical knowledge for teaching after participating in the course, considering their specialisation backgrounds and initial knowledge levels?

Exploring this question in the context of program design has the potential to yield valuable insights for creating large-scale, accessible programs that enhance the skills and knowledge of in-service teachers, who have less free time to attend face-to-face TPDs. Ultimately, the research aims to guide online TPD developers in considering critical factors that ensure a course is inclusive for all teachers, regardless of their initial knowledge levels. This includes strategies for organising course content to accommodate diverse backgrounds and implementing timely supports that help less-prepared teachers access and utilise the knowledge provided.

Professional Development Programs for Mathematics Teachers

To enhance the quality of mathematics instruction, teacher professional development (TPD) programs are commonly utilised to achieve this goal, as highlighted by Jacobs et al. (2017). Ensuring the effectiveness of TPD programs involves two main research and development directions: designing the programs and evaluating them. Borko et al. (2014) suggested that TPD models can be classified on a spectrum from highly flexible to highly structured. In other words, TPD models can vary in the degree of flexibility they offer for implementation and adaptation across different contexts. Some models may offer greater room for customization and adaptation, while others may follow a more rigid, standardised structure.

A TPD model can also offer different modalities of participation, for instance: in-person, where teachers gather for workshops, and e-learning in which participants interact with others and content through a digital Learning Management System (LMS). LMSs give educators the flexibility to engage in TPD activities at their own pace from any geographical location or via blended mode that combines in-person and e-learning learning activities. TPD digital modalities offer flexible learning and reflect the evolving landscape of professional development in response to the growing global access to technology and its integration into teacher professional development (Burns, 2013). Technology alone does not guarantee the effectiveness of TPD, and ensuring quality experiences should still be considered to support effective design and implementation principles (Powell & Bodur, 2019).

Researchers have sought to identify the key characteristics that influence the effectiveness of TPD programs (Meyer et al., 2023). Garet et al. (2001) recommended analysing programs based on two groups of characteristics: structural features, such as the type of activities, duration, and collective or individual participation; and key characteristics, including the content addressed, learning methodology, and coherence with teacher beliefs and policies. Desimone's (2009) review emphasised that effective professional development should focus on subject matter content, involve active teacher participation, align with teacher beliefs and policies, and encourage collective participation. Despite relying largely on non-experimental, teacher self-reported data, Desimone's identified features are widely recognised as fundamental design elements for effective professional development. In alignment with others, Rosli and Aliwee (2021) also highlighted that the core features are useful if they include content knowledge, exciting opportunities for teachers to learn, and coherence with other PD activities.

In the context of professional development programs for mathematics teachers, Copur-Gencturk et al. (2019) identified key activities that foster greater development of mathematical knowledge for teaching. These include activities focused on knowledge of mathematical content and strategies specific to content implementation, with an additional emphasis on curriculum content knowledge and student productions. Moreover, TPD programs centred on specific mathematics teaching themes allow teachers to enhance their skills and teaching practices in that specific theme (Dash et al., 2012; Oslund, 2016; Pang, 2016; Rosli & Aliwee, 2021). Such an improvement in teaching practices positively affects students' mathematical performance (Niess & Roschelle, 2018). However, as reported by Rosli & Aliwee (2021) in a systematic literature review, certain attitudinal factors of teachers play an influential role in the transition from effective teaching practices to students' achievement—a key indicator of TPD effectiveness. Regarding the type of TPD program, such as online courses, seminars, discussion groups, and observation-based monitoring, Rosli and Aliwee (2021) pointed out that there was no conclusively more effective type than another. Goos et al. (2020) crafted a blended TPD program tailored for out-of-field teachers in Ireland, rooted in essential principles of mathematics education. Emphasising robust MKT and the integration of content knowledge (MCK) and pedagogical content knowledge (PCK), this blended learning approach was reported to be successful not only to support the development of teachers' MCK and PCK but also to enhance their adaptability and identity formation as in-field mathematics teachers. Similarly, Nel and Luneta (2017) argued that there is no one-size-fits-all effective program for all teachers, as the key to success lies in their participation. In other words, different types of programs could facilitate teachers' participation according to their possibilities and specific contexts.

Theoretical Framework: Mathematical Knowledge for Teaching

The teaching of mathematics is a complex activity that requires various skills and knowledge from teachers (Ball, 2003; Ball et al., 2005; Ball et al., 2008). Building on the work of Shulman (1986), who defined pedagogical content knowledge as the intersection between content and pedagogy specific to the task of teaching. Ball et al. (2008) proposed a comprehensive conceptual model of Mathematical Knowledge for Teaching (MKT) to characterise different types of pedagogical and disciplinary knowledge necessary for organising and managing appropriate teaching processes (Ball et al., 2008). Simply put, Ball et al. (2009) defined MKT as "the mathematical knowledge needed to carry out the work of teaching mathematics" (p. 96)." Some studies have shown the relationship between teachers' MKT and the quality of instruction (Hill et al., 2005; Campbell & Lee, 2017). Consequently, there have been attempts to design PD programs that enhance teachers' mathematical knowledge for teaching, enabling them to promote greater student thinking and reasoning in mathematics lessons (Jacobs et al., 2017).

As shown in the diagram in Figure 1 (Ball et al., 2008), on the left side of the oval, three components of content knowledge are distinguished as subject matter knowledge for mathematics teachers. These encompass strictly mathematical issues that teachers need to enhance their classroom practice and teaching. "Common content knowledge" corresponds to mathematical knowledge that most people have, for example, knowing how to solve a division of natural numbers. "Specialised content knowledge" is related to specific knowledge for the teacher's role, for example, distinguishing the underlying properties that justify a rule for dividing natural numbers or explaining through a bar diagram an algorithm for dividing two numbers. "Horizon content knowledge" corresponds to knowledge of how different content areas relate throughout the curriculum. Among these three types of subject matter knowledge, an unclear relationship between the nature of "Horizon content knowledge" and teaching has been reported (Jakobsen et al., 2012). While Ball et al. (2009) described it as a form of mathematical peripheral vision necessary in teaching, providing an awareness of the broader mathematical landscape teaching requires, this component has been somewhat overlooked (Mosvold & Fauskanger, 2014).

On the right side of the oval (in Figure 1), the required pedagogical content knowledge for teaching mathematics is detailed. These are knowledge components that, in addition to mathematics, involve aspects of teaching and learning. "Knowledge of content and students" refers to understanding the characteristics of students at different levels where a specific mathematical content is taught. Continuing with the example of the division of natural numbers, it involves knowing the types of errors that students frequently make when calculating a division, to be able to address them productively. The purpose of anticipating errors is not to prevent them but to seize them as valuable learning opportunities, in line with a constructivist perspective. "Knowledge of content and teaching" relates to a combination of knowledge about teaching and knowing about mathematical content. This encompasses sequencing content for instruction, selecting appropriate examples, evaluating instructional methods, and making informed decisions during classroom discussions. For example, when and how to use concrete materials (such as base-10 blocks) to teach division. Finally, "Knowledge of content and curriculum" refers to understanding the curricular evolution of content, for example, knowing which aspects of division are studied in the 3rd grade (aged 8 to 9 years old), the type of algorithms expected from children at this level, as well as the types of problems they are expected to solve.

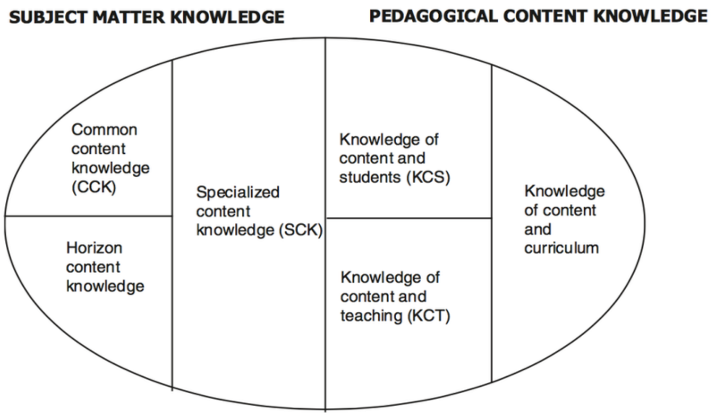

Figure 1. Categories of MKT, Ball et al., 2008 (p. 403).

An Online Teacher Professional Development Program: The Case of Suma y Sigue

Suma y Sigue, or *Add and Follow* in English, is an online TPD program as e-learning developed by a multidisciplinary team of teachers, mathematicians and experts in education at a mathematics research centre at the University of Chile (Martínez et al., 2020). The program is designed to enhance teachers' MKT through contextualised learning activities that encourage a thorough analysis of curricular mathematical content. The program is guided by three key principles: (1) a constructivist perspective of learning, emphasising that knowledge emerges as teachers solve mathematical and didactic problems, requiring them to utilise prior knowledge and restructure it to find a solution (Thompson, 2014), (ii) the importance of realistic contexts, deeming it fundamental to present mathematical and didactic problems within plausible contexts, enabling teachers to comprehend mathematics and its teaching (Freudenthal, 2012), and (iii) alignment with the MKT model, indicating that teaching mathematics necessitates specific knowledge that can be distinguished, characterised, and developed (Ball et al. 2005, 2008).

The program encompasses 16 online courses tailored for primary, upper primary school and secondary school teachers in Chile. Each course delves into a specific curriculum topic, concentrating on the development of different domains of MKT (Ball et al., 2008). Teachers can participate in different courses of the program based on their individual needs, and there is an intentional specificity for each course. Some courses primarily emphasise subject matter knowledge, especially those addressing topics from more advanced school levels, while others integrate aspects of pedagogical content knowledge. The courses include both asynchronous and synchronous activities. This design allows teachers to engage in activities at their convenience, with some synchronous sessions dedicated to deepening content through direct interaction with a TPD facilitator and peers. The TPD facilitators are expert mathematics teachers who are trained to guide the discussion based on the asynchronous activities in synchronous sessions. The duration of each course is approximately 40 hours, with an allocated time frame of 11 weeks for completion. Approximately 80% of the course duration is devoted to asynchronous activities, delivered via an online learning platform. These asynchronous activities are contextualised and structured sequentially, presenting a storyline centred around a scenario that introduces a mathematical or didactic problem. This narrative guides the tasks required to address various domains of MKT, as elucidated by Ball et al. (2008). The asynchronous activities are designed to promote autonomous learning, but tutors answer participants' questions with a short waiting period. Every course comprises 4 to 6 online workshops, each containing 5 to 6 activities. Each activity delves deeply into a specific theme of the mathematical content, and tasks unfold as the storyline progresses. Issues or conflicts emerge in the development of the story in a nonlinear manner, providing the opportunity to adjust the didactical focus of the task or change the type of knowledge initially addressed. This design is preliminary used to promote exploration, critical thinking, and problem-solving skills, as learners navigate through a variety of interconnected scenarios and challenges that contribute to a more holistic understanding of the subject matter (Martínez et al., 2020). For example, in a scenario involving a discussion between two characters, a conflict arises around a mathematical idea, prompting learners to respond to a series of questions that encourage them to unpack the mathematical concepts that surfaced during the conflict. The synchronous discussion sessions are structured to deepen some of the course content, for instance, by connecting them to a teaching situation. Before the session commencement, an "activation" phase occurs, during which participating teachers individually reflect on a mathematical or didactic situation proposed as part of the course activities. This ensures that all participants are prepared for the subsequent "synchronous discussion," where they convene in an online meeting through a platform to discuss the situation, fostering collaborative engagement. Following the synchronous discussion sessions, there is a "discussion projections" stage, taking place afterwards. In this phase, teachers reflect on the learning outcomes from the discussion and further explore the topic through new questions posed in a virtual forum.

This sequential learning environment, where tasks and content progressively unfold, facilitates not only an integrated approach to various types of knowledge but also a scaffolded learning process (Meyer et al., 2023). To track the trajectory of teacher learning throughout each course, a comprehensive set of four assessment tools is provided at the end of each module of activities. Participating teachers are required to take a pre-test at the beginning of the course and complete each assessment at the end of each module to progress to the subsequent module.

Multiplication and Division: A Course from the Suma y Sigue Program

This course addresses the mathematical knowledge for teaching multiplication and division of whole numbers. It was designed for teachers teaching mathematics in Grades 2 to 4 (aged 7 to 9 years old) of primary school in Chile. These teachers should have graduated from a primary school education program. For context, primary school education programs, whether or not they explicitly mention mathematics, include four courses dedicated to mathematics and its teaching methods. Programs with a specialisation in mathematics encompass approximately six additional courses in this subject (Valenzuela & Martínez., 2016). To be mathematics specialists, teachers can also obtain a postgraduate certificate focused on a particular educational level, such as teaching mathematics from 5th to 8th grade. Obtaining such certification usually involves approximately 700 classroom hours spread over two years. It is important to note that the course content addressed in this study is commonly found in primary school education programs, with or without specialisation in mathematics (Valenzuela & Martínez, 2016). The course duration is three months, which means participants can start and complete all activities and assessments within that time.

As depicted in Figure 2, the course consists of four asynchronous activities and two synchronous collective discussion sessions. This participant-oriented design, while allowing participants to engage in activities at their convenience, also establishes a structured framework that guides them consistently throughout the course. The course covers both dimensions of MKT: subject matter knowledge and pedagogical content knowledge.

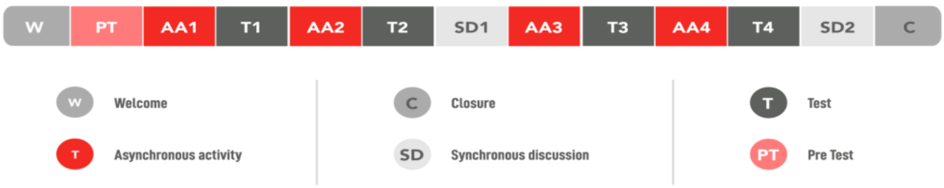

Figure 2. Structure of the Working with Multiplication and Division Course.

The course begins with a synchronous welcome session where the *Suma y Sigue* program is introduced. During this session, the course objectives, activities, evaluation methodology, and work schedule are presented. The course is structured into four asynchronous activities, each focusing on distinct aspects of MKT for teaching multiplication and division of whole numbers to primary students. In the first asynchronous activity, the exploration of multiplicative situations takes centre stage, involving the analysis of diverse problem types and their representations. This foundational knowledge is essential for framing various interpretations of whole number multiplication and division, guiding instruction through problems that imbue these operations with meaning. Moving into the subsequent asynchronous activities (2 and 3), a deep dive into the algorithms and calculation strategies associated with multiplication and division unfolds. The properties of these operations and calculation strategies are elucidated through generic examples emerging from the analysis of contextual situations.

The final asynchronous activity (4) focuses on problem-solving, delving into crucial aspects for effective classroom implementation. This activity incorporates elements of common content knowledge, specialised content knowledge, knowledge of content and teaching, and knowledge of students and content. For instance, common errors in setting and solving problems are analysed, showcasing how these missteps can be leveraged as opportunities to enrich learning and enhance the interpretation of mathematical results. Table 1 provides a comprehensive overview of the contents and components of MKT addressed within each module of the course.

Table 1
Activities and their features in the modules of the course

No.	Asynchronous activity	Features	MKT Categories Involved
1	Multiplicative Situations	Multiplicative problems involving grouping, combination, and two-dimensional arrays. Division as a solution to grouping problems.	Common and specialised content knowledge; Knowledge of content and teaching of multiplication and division.
2	Multiplication	Justification of the conventional multiplication algorithm based on the properties of this operation. Construction of multiplication tables and various multiplication calculation strategies.	Common and specialised knowledge of multiplication. Knowledge of content and students.
3	Division	Justification of the conventional division algorithm based on the properties of this operation. Various division calculation strategies.	Common and specialised knowledge of the division.
4	Problem-solving	Problem-solving, analysing relevant aspects of this type of activity in the classroom, such as the use of representations, understanding different solution strategies, and interpreting results.	Common knowledge of division and multiplication. Knowledge for content and teaching; Knowledge of content and students.

In the synchronous discussions, participants engage in in-depth discussions facilitated by a TPD facilitator, exploring the topics covered in each asynchronous activity. In the first session, the discussion delves deeper into the teaching of multiplicative situations, specifically focusing on grouping and combination, while analysing associated models and representations of related tasks. The second session centres on discussions about teaching various division strategies and using representations.

Additionally, this course includes four assessment tests conducted at the end of each module. The assessment tests are used as the data collection instrument in this study to evaluate the content associated with various components of MKT (here, teachers' performance in MKT) addressed through asynchronous activities and synchronous discussion sessions.

Methodology

The study recruited primary mathematics teachers from public or subsidised schools, resulting in two groups with different characteristics. Due to non-random assignment, a quasi-experimental design with non-equivalent groups was employed, while both groups underwent the same online TPD course. In this design, there was no control group; instead, the pre-existing groups were differentiated by a specific factor that distinguishes them (Shadish et al., 2002). This design allows for examining intervention effects while acknowledging initial group differences and attempting to control for potential confounding variables.

Participants

In total, 142 primary school teachers enrolled in the online TPD course during 2021. These teachers voluntarily joined the course from various regions in Chile. Among the initial 142 applicants, 28 teachers officially notified that they could not continue and withdrew before completing the course, and 15 teachers decided not to participate in the study, opting out of the pre- and post-tests. Therefore, 99 teachers were considered participants in the study. Among them, 54 teachers held a university certification specialising in mathematics teaching, which could have been obtained during initial or teacher training or through a postgraduate course. The remaining 45 teachers indicated no specialised education in mathematics. The term "specialisation" here refers to the nature of the courses they took to become teachers. Due to the course registration policy, access to participants' demographic information was limited. However, the majority of the participants were female teachers (n = 88, 89%), which aligns with the fact that approximately 78% of primary school teachers in Chile were women (Ministry of Education [MINEDUC], 2022).

The Instrument

To assess teachers' performance in MKT, five data collection instruments were used: a pre-test administered before the course and four tests conducted after each module of activities during the course. Each test had seven items that evaluated understanding gained in each module, emphasising different domains of MKT related to the instruction of multiplication and division of natural numbers.

As shown in Table 2, each test comprised seven items: four multiple-choice questions and two true/false statements. The tests were developed by a team of experts in item design within the theoretical framework of Mathematical Knowledge for Teaching (Ball et al., 2008), with a focus on number instruction. The team created a table of specifications that linked the main components of the MKT model with the course content (multiplicative situations, the study of multiplication as an operation, the study of division, and the solving of multiplicative problems). Subsequently, an external reviewer examined the items using two criteria: (1) their psychometric quality, according to the Standards for Educational and Psychological Testing (AERA, APA & NCME, 2014), and (2) the empirical results obtained from pilot administrations of previous versions of the items. This review and revision process led to the final version of the tests used in the present study.

The tests were available on the platform for 72 hours, with a uniform timeframe for all teachers. Before undertaking any activity or test on the platform, teachers were required to log in with their personal username and password and sign an agreement committing not to engage in any form of academic misconduct or unethical practices. Once initiated, teachers had 90 minutes to complete each test. The pre-test consists of nine items: seven multiple-choice questions and two true/false statements. The pre-test follows a similar approach to the course tests. During the post-test phase, participants responded to these nine items again throughout the course. These items were seamlessly integrated into the course’s tests (2–3 per test) based on their alignment with the respective module themes, so teachers did not recognise them as repetitions of the pre-test items.

Table 2
Domains of MKT Assessed by the Tests

Instrument	Assessed MKT Domain	No. Items
Test 1 (T1)	Common content knowledge: solving multiplicative problems.	3
	Specialised content knowledge: formulating multiplicative problems, understanding reasoning in solving a multiplicative problem.	2
	Knowledge of content and teaching: using representations in solving multiplicative problems.	2
Test 2 (T2)	Common content knowledge: identify properties in the Pythagorean table, identify ways to express a number multiplicatively.	3
	Specialised content knowledge: identify properties that justify a multiplication calculation procedure, and evaluate the validity of a multiplication calculation procedure.	3
	Knowledge of content and students: describe possible errors in a multiplication calculation	1
Test 3 (T3)	Common content knowledge: solving multiplicative problems, describing a division procedure.	4
	Specialised content knowledge: identifying properties that justify a calculation procedure, evaluating the validity of a division calculation procedure, formulating division problems.	3
Test 4 (T4)	Common content knowledge: solving multiplicative problems, describing a division procedure.	5
	Specialised content knowledge: interpreting the solution of intermediate steps in solving a problem.	1
	Knowledge of content and teaching: using representations in solving multiplicative problems.	1

Analysis

The analysis comprised data preparation and the primary analysis to address the research questions. The process of analysis was carried out in three stages, detailed as follows:

Preparation of the data

In the first phase of the data analysis, the responses to all items were scored in each of the four tests dichotomously as correct (2) or incorrect (1). To assess the psychometric properties of the instruments and items, learner responses to individual test items were examined, evaluating both the quality of the items and the overall test. Calculations were made in regard to difficulty, discrimination indices for each item, and reliability (Cronbach's Alpha) for each one of the four tests. Item difficulty and discrimination were only calculated for multiple-choice questions. The mean item difficulty for each test ranged from .36 to .64. The mean item discrimination for each test ranged from .25 to .43. Cronbach's Alpha values for each test ranged between .40 and .60. The scores were deemed satisfactory (D'Sa & Visbal-Dionaldo, 2017; Taber, 2018).

Teacher MKT performance between two groups with different specialisation backgrounds

To address the research question, first, the differences in the pre-test results among teachers with a specialisation in mathematics (Math+) and those without a specialisation (Math-) were examined. The mean value of teachers' re-responses was calculated for all pre-test items throughout the course. These responses, obtained during the participants' second attempt, were then treated as a post-test. To examine differences between the Math+ and Math- groups during their participation in the program while controlling for pre-existing knowledge differences, an analysis of covariance (ANCOVA) was performed. This analysis assessed differences among the two groups in changes in post-test knowledge scores across all four tests (T1, T2, T3, T4) separately.

Tracking change in teacher MKT Performance with initial knowledge levels

Dependent samples *t*-tests were conducted to assess within-group differences. Effect sizes were calculated using Cohen's d, as outlined by Cohen (1988). Cohen's guidelines for interpreting effect sizes are considered as small = 0.2, medium = 0.5, and large = 0.8. The last stage aims to unravel how performance evolves throughout the course under the influence of these combined factors. To initiate this exploration, the results are presented descriptively and visually, providing a comprehensive overview of the observed patterns. To investigate the associations among the tests (pre-test, T1, T2, T3, T4, and post-test) while controlling for group membership (Math+ and Math-), partial correlation analyses were performed. These analyses allowed us to control for the influence of group participation, isolating the relationships among the test scores.

To account for teachers' initial knowledge levels, they were classified based on their initial knowledge performance. To analyse the score distribution, the Shapiro-Wilk normality test was conducted. The results showed a normal distribution (M = .46, SD = .18), with a minimum of .11 and a maximum of .89. To classify participants based on their initial performance, three classes were defined using the pre-test mean ± 1/2 *SD* as cutoff points. Class 1 (N = 38) included teachers whose pre-test scores were below the mean minus 1/2 *SD*. Class 2 (N = 28) included those with pre-test scores at the mean ± 1/2 *SD*, and Class 3 (N = 33) included teachers whose initial performance was above the mean + 1/2 *SD*. Considering ±1/2 *SD* ensured equal numbers of participants in each class. Table 3 shows the number of teachers classified in each group.

Table 3

Number of Teachers Based on Their Background Profiles

	Class 1	Class 2	Class 3	Total
Math -*	20	12	13	45
Math +**	18	16	20	54
Total	38	28	33	99

*Math- indicates the group of teachers without specialisation backgrounds in mathematics.

**Math+ indicates the group of teachers with specialisation backgrounds in mathematics.

Results

Teacher MKT Performance Among Groups During the Course

A comparison of pre-test, post-test, and performance during the course among the Math+ and Math- groups is presented in Table 4. The Math+ group showed higher mean scores across all time points compared to the Math- group.

While the Math+ group (Mpre = .48, SD = .18) performed better in the pre-test compared to the Math- group (Mpre = .44, SD = .17), this difference was not significant, $t(97)$ = 1.24, p = .24. After adjusting for pre-test scores, the ANCOVA results revealed that the difference between two groups in their post-test was statistically significant just at .05 level; ($F(1, 96)$ = -4.13, p = .045<.05). Additionally, while no noteworthy differences between Math+ and Math- groups were observed at T1, T3, and T4, the results at T2 showed that the Math+ group significantly outperformed Math- group.

Table 4
Comparison of post-test and the performance during the course between groups

Groups	Post-test	T1	T2	T3	T4
Math +	.67(.18)	.73(.16)	.78(.16)	.75(.18)	.76(.20)
Math -	.58(.18)	.66(.16)	.63(.15)	.69(.19)	.67(.22)
F(1, 96)	4.13*	2.95	22.10**	1.86	3.50

* p-value at .05
** p-value at .01

Change in Teacher MKT Performance Across Initial Knowledge Levels During the Course

These results in three stages show how TPD impacts teacher MKT over time, considering both initial conditions and ongoing growth. First, within-group differences were calculated for each group using paired sample t-tests. These tests revealed a significant difference among pre-test and post-test scores for participants in each group, separately, at the .01 level. Specifically, MKT performance in the Math+ group increased from (Mpre = .48, SD = .18) to (Mpost =*.67*, SD = .18), indicating a statistically significant change (t(53) = 5.40, p<.01). Similarly, the Math- group showed an increase from (Mpre = .44, SD =.17) to (Mpost = .58, SD = .18), also reflecting a statistically significant change (t(44) = 4.40, p<.01). The calculated Cohen's d was slightly more than 0.50 for both the Math+ and Math- groups, indicating a medium effect size for the pre-post difference in both groups. The medium effect size suggests a notable and meaningful change in teacher MKT over the course of TPD.

Second, the partial correlation analysis examines how different test scores (pre-test, T1, T2, T3, T4, and post-test) are interrelated, considering the influence of group membership (Math+ and Math-). As shown in Table 5, T1, T2, T3, and T4 have strong correlations with the post-test. These correlations suggest that, in general, participants' performance on these tests is closely related to the post-test. Regarding correlation with pre-test performance, the results indicate that T2 and T3 scores are significantly correlated with pre-test scores. However, these correlations are not as strong as those observed with T1 and T4. Notably, the weakest correlation among the tests is between T2 and T3, despite both tests including questions that focus on the justification of calculation procedures using the properties of multiplication (T2) and division (T3).

Table 5
Partial Correlations among test scores, controlling for Math+ and Math- membership

Correlation	Pre-Test	T1	T2	T3	T4	Post-Test
Pre-Test	1					
T1	.41**	1				
T2	.21*	.35**	1			
T3	.21*	.43**	.24*	1		
T4	.39**	.52**	.34**	.38**	1	
Post-Test	.39**	.73**	.50**	.70**	.50**	1

* p-value at .05
** p-value at .01

According to these correlations, the changes in participants' performance throughout the course can be considered as a trajectory of MKT development for each group. This suggests that the sequence of test scores reflects meaningful progress in MKT over time, enabling us to track participants' knowledge and skills throughout the course.

Third, the performance patterns for teachers' performance across different classes, categorised by low, medium, or high initial MKT, and within each group, distinguished by their mathematical specialisation profiles (Math-, Math+), are illustrated in Figure 3. Similar patterns are observed in teachers' performance across classes with low and high initial MKT levels in both groups at T1, T2, T3, and T4. However, the participants categorised as having a medium level of initial MKT exhibited quite different performance throughout the course. This is consistent with the lower partial correlations of T2 and T3 with the pre-test, as well as the significant difference in achievement T2 between Math+ and Math-, as shown in Table 2. For a medium level of initial MKT, having a specialisation makes a difference in T2 performance.

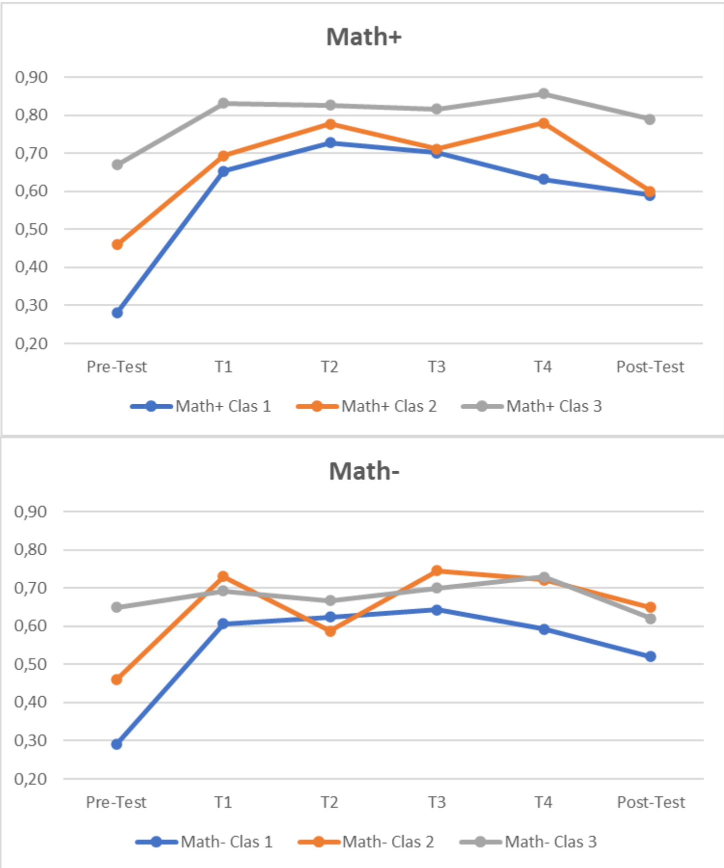

Figure 3. Trajectories of performance in MKT across initial knowledge levels during the course.

Discussion and Conclusion

This study delves into the specific realm of teaching multiplication and division of whole numbers, offering a microcosm of broader challenges and opportunities within mathematics pedagogy. The results indicate that the intervention was successful for both the Math+ group (teachers with maths specialisation) and the Math- group (teachers without maths specialisation), with the Math+ group showing a slightly greater improvement. Martínez et al. (2020) highlight the successful design and implementation of the online TPD program. The following sections discuss the main findings and their implications in relation to the study's objectives.

Effectiveness of the Course and Differences Among Teacher Groups

Notably, the result highlights the potential of such initiatives to address the global challenge of equipping teachers with the knowledge to teach mathematics (MKT) effectively, as noted by Ball et al. (2008), particularly in contexts where specialised mathematics teachers are scarce. It is interesting to note that both groups, with and without a specialisation in mathematics, showed similar initial MKT knowledge of multiplication and division before the course began. The similar initial MKT knowledge between the two groups can be attributed to the fact that the common curriculum for primary education teachers covers mathematical knowledge in the first four grades, such as multiplication and division of whole numbers (Valenzuela & Martínez, 2016).

Specialisation courses are usually focused on mathematical content from more advanced courses (upper primary or 5th to 8th grades, students aged 10 to 13 years old); thus, they did not yield specific advantages or difficulties compared to the other group in most course assessments. Indeed, mastering or understanding advanced content does not necessarily imply a greater MKT proficiency in elementary mathematics. Considering the tests during and after the course, and the significant results at T2 and in the post-test, the intervention was beneficial for both groups, with a more pronounced effect on teachers with a mathematics specialisation. Delving deeper into the differences, the test T2 mainly assessed the recognition and use of the properties of multiplication in calculation procedures, a subject that is probably discussed in greater depth in more advanced specialisation courses, given its relationship with Algebra. This difference between the two groups in the test T2 might be due to the specialised training and more profound content knowledge that Math+ teachers possess, enabling them to implement the intervention more effectively and understand the material more deeply, as described by Martínez et al. (2020). In total, the results are aligned with the broader educational challenges and opportunities discussed in the literature on mathematics teaching and learning (e.g., Goos et al., 2020; Wiggan et al., 2021).

Understanding Within-group Improvements and Course Accessibility

The within-group improvements provide a basis for understanding subsequent performance correlations, which, in turn, validate the meaningfulness of these improvements in contributing to overall MKT development trajectories. All the assessments evaluate content at varying levels of difficulty; however, comparisons between them are not possible. The partial correlations highlight the relationship among all of the assessments. It is challenging to definitively determine whether the improvement in teachers' performance in MKT stems solely from the course activities. Moreover, across the three classes based on their initial levels of knowledge, the overall performance in the course showed a similar pattern, especially for participants with low and high initial MKT levels in both groups. However, the pattern was different for participants at the medium level of initial MKT, particularly in the Math- group. Notable differences occurred during T1, T2, and T3, with T1 and T3 showing slightly better performance than Class 1, and T2 showing slightly lower performance than Class 3. This aligns with the complex knowledge domain identified in teachers without adequate preparation to teach mathematics, which demands specific mathematical knowledge and an understanding of learning characteristics (Carrillo et al., 2018). To draw a secure inference, the results underscore the idea that the course characteristics made it accessible to teachers with different backgrounds and knowledge bases; however, we agree with As noted by Nel and Luneta (2017), there is not a single program that is effective for all teachers, rather the effectiveness of any program depends on teachers' participation. In Chile, there is differentiated training for pre-service primary education teachers (Chandía et al., 2021; Valenzuela & Martínez, 2016) which manifests in highly variable mathematics achievement at the end of teachers' initial training (CPEIP, 2022), a TPD program such as *Suma y Sigue*, which is deployed nationally and available free of charge to a large number of teachers in Chile, needs to be accessible to teachers with different knowledge bases. The results obtained provide evidence that the course analysed is accessible to teachers, offering equitable learning opportunities to develop MKT.

Implications and Limitations

Taken together, the results reinforce theoretical propositions that connect teachers' opportunities to engage with content-specific tasks and reflective discussion to the development of MKT (Ball et al., 2008). Consequently, the study supports models of TPD that integrate conceptual focus, collaboration, and reflection as mechanisms for professional growth (Desimone, 2009; Borko et al., 2014). According to Copur-Gencturk et al. (2019), courses targeting MKT development have proven effective in enhancing teachers' knowledge. TPD programs in Chile, however, often exhibit limited efficacy in enhancing pedagogical knowledge (Carrasco-Aguilar et al., 2023). Therefore, the results of this study provide important insights into the trajectory of change in MKT performance among teachers and the efficacy of such courses. Specifically, the study highlighted that certain course design features may be relevant to explaining the results. The course was highly structured, as suggested by Borko et al. (2014). Its features offer an active learning experience, in which teachers not only "receive" information but also engage in analytical efforts related to didactic and mathematical aspects. Explanations about expected answers were provided subsequently. Secondly, this course integrated individual work with group discussions, in which teachers could raise questions by conversing with monitors and fellow educators. These elements align with the findings of Powell and Bodur (2019), who reported that teachers participating in an online TDP on social science teaching noted that a less active experience, coupled with a lack of opportunities for real conversations, limited their learning. Another study on the effectiveness of an online TDP course, similar to the present study, focused on specific pedagogical content in mathematics and combined collective discussion with individual work, found that teachers experienced changes in both their knowledge and classroom practices (Dash et al., 2012). The course's design aligns with the findings of Bragg et al. (2021), who highlighted that effective online professional development programs for teachers emphasise practical learning activities, reflection, and the relevance and application of knowledge and skills.

Three primary limitations identified in the study may be useful to guide future research. Firstly, the participation rate of those who completed the course is an important factor for such a study. As highlighted by Nel and Luneta (2017), participants' active engagement is a critical determinant of success for any TPD program. While presenting results from the current sample size is valuable, future studies should examine the reasons behind dropout rates and the factors influencing those who choose to continue the course but refrain from active participation in assessments. Secondly, another limitation of this study is the absence of a control group, as it is generally challenging to find teachers with similar characteristics to those taking the course who are willing to undergo assessments without directly benefiting from the training. Third, the use of closed-ended test items may have restricted the depth of participants' responses, potentially limiting insights into their reasoning processes and MKT. Despite this limitation, it can be argued that the changes observed were due to teachers' participation in the course, as assessments designed to evaluate teachers' performance in MKT showed significant increases. It would be interesting to explore in the future whether the increase in knowledge resulting from the course is sustained over time.

Conclusion

Focusing on teaching multiplication and division of whole numbers may seem like a narrow aspect of mathematics education. Findings from this study, however, serve as an illustrative example of the broader potential impact of online professional development programs on educators' learning and professional growth, as underscored by Borba and Llinares (2012). Particularly noteworthy is the effectiveness of such programs for teachers with diverse backgrounds, including teachers with and without adequate preparation to teach, as suggested by Goos et al. (2020).

This finding resonates beyond Chile's local context, as the shortage of specialised mathematics teachers is not unique to this country. Given that elementary school teachers in Chile often exhibit insufficient mastery of mathematical and didactic knowledge upon completing their initial education

(CPEIP, 2022), it is important to offer practical and inclusive TPD programs to the community, which reach a large number of teachers throughout the country, are accessible to a diverse group of teachers, and provide real opportunities to improve their MKT. Offering such TPD is also in line with Ball et al.'s (2008) research, as this knowledge goes beyond general mathematical proficiency and involves a specialised understanding of mathematical ideas as they are used in teaching practice and are particularly useful for in-service teachers (Ball et al., 2008).

The results of the *Suma y Sigue* program align with the literature that acknowledges the online format of TPD programs for teachers' professional growth, especially for those in geographically isolated areas (e.g., Burns, 2013; Meyer et al., 2023). Such a program, particularly beneficial for a large country like Chile, has the potential to provide similar opportunities for all teachers, including those in remote areas who often lack access to them in their regions. Access to online programs that strengthen pedagogical knowledge and skills can improve mathematical education and enable all teachers to benefit. The results support the idea that online professional development can be a flexible and effective way to enhance teaching performance in a geographically and administratively diverse context, such as that of Chile.

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Ethical approval

Ethical approval for the research was granted by the Faculty of Physics and Mathematical Sciences Ethical Committee, Universidad de Chile. Informed consent was given by all participants for their individual anonymized data to be published.

Competing interests

The authors declare there are no competing interests.

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