Mathematics Teacher Education and Development

2001, Vol. 3, 62-73

Defining as a Vehicle for Professional Development of Secondary School Mathematics Teachers

Roza Leikin
The University of Haifa

Greisy Winicki-Landman
Oranim – School of Education of the Kibbutz Movement

This paper describes a professional development program for secondary school mathematics teachers. The issue of mathematical definition is the focus of the program. The program aimed to develop teachers' subject matter knowledge including knowledge of substance of mathematics and knowledge about the nature and discourse of mathematics. The rationale of the course is described as well as some outcomes of its implementation. The activities included in the program deal with some mathematical concepts and different didactical approaches to secondary school mathematics. The two activities presented in this paper, exemplify some of the ideas.

Mathematics teachers make critical decisions about the mathematics they teach and the way they teach mathematics. This requires teachers to be aware of reform-oriented approaches for teaching mathematics as well as to be mathematically educated. Research in mathematics teacher education shows that teachers' mathematical knowledge must be deep and robust in order to make intelligent decisions in the course of teaching mathematics (Ball, 1992, 1997; Ma, 1999). Thus, it is important that teacher educators provide teachers with a range of opportunities to deepen their content knowledge and to explore multiple instances through which a new vision of mathematics and teaching mathematics is presented (Ball, 1992).

The structure of a mathematical curriculum may be described in terms of objects and operations with these objects (Schwartz, 1999). This structure is built upon a foundation constructed of definitions, axioms, theorems, and proofs connected by logical rules. As each building cannot be stable without a robust foundation, the building of individual mathematical knowledge should be grounded in deep and robust understanding of the essence and role of meta-mathematics. Ball (1992) stated that some of the most significant teachers' decisions are connected to guiding the direction and balance of classroom discourse. At the same time, Sfard (2000) points out that defining and proving are basic units of mathematical discourse in its broad sense. The community of mathematics educators has a long tradition of discussing the mathematical and cognitive issues of the proof process (Balacheff, 1987; Chazan, 1993; De Villiers, 1990; Hanna, 1989). However, the issue of definitions in mathematics, as well as in mathematics education, is less exposed in educational research (De Villiers, 1998; Vinner, 1991). This paper was motivated by the need to further develop and investigate mathematical and didactical aspects of the defining process in classroom discourse.

We accept an assumption that teachers' personal learning experiences influence their teaching practices (Cooney, 1994; Comiti & Ball, 1996).

Correspondingly, one of the broadly-accepted ways for development of teachers' mathematical knowledge is in-service programs in which teachers engage in different learning experience and become mathematics students themselves (Borasi, 1999; Cooney, 1994). These programs for professional development of mathematics teachers should be directed towards the enhancement of different domains of teachers' knowledge (Wilson & Shulman, 1987). In our program the issue of definition was chosen to develop teachers' subject-matter understanding, their knowledge of students, and their ability to manage learning. Discussing with the teachers 'what is a definition' and 'how to define' allows focusing on (a) *logical relationships* between mathematical statements, (b) *didactical sequences* of learning, (c) *mathematical connections*, and (d) *mathematical communications*. The learning settings in the program vary in a way that allows development of an assortment of teaching approaches and methods that teachers could apply in their classroom. In this way, the teachers might be exposed to multiple instances that present mathematics and its teaching in new ways.

Ball (1992) defines four dimensions of teachers' subject matter understanding. The issue of definition can be used to enhance teachers' knowledge in these four dimensions. Knowledge of substance of mathematics is developed by discussion of different ways to define a particular concept, making the understanding of the concept deeper and more flexible. This kind of discussion allows us to elaborate issues, such as mathematical correctness, mathematical meaning and mathematical connectedness. The second dimension – knowledge about the nature and discourse of mathematics – is in our context mainly connected to the discussion of the role of justification in mathematics, and to debate regarding convention versus logic in the development of mathematical ideas. The discussion of the evolution of some mathematical ideas (definitions) across history (see for example, Kleiner, 1989) was directed towards the enhancement of teachers' understanding according to Ball's third dimension, knowledge about mathematics in culture and society. Finally, variations in the learning settings in our program, and the topic of arbitrariness of definitions, were intended to develop capacity for teachers' pedagogical reasoning about mathematics.

The Structure of the Program

Our professional development program is for in-service secondary school mathematics teachers and includes eleven different mathematical activities focused on the issue of definition. Each activity deals with the concept of mathematical definition, as well as with different definitions of a mathematical concept. The teachers, who took part in the activities were involved in discussions about the logical relationships between different mathematical statements related to a certain concept. Along with De Villiers (1998), we aimed to develop teachers' awareness that alternative definitions of a concept are possible. For a specific mathematical concept, different mathematical statements were chosen. Some of these were *equivalent* and the others were *competing* or *following*[1] (for details see Winicki & Leikin, 2000).

Defining[2] can be considered as naming, but in fact is much more than naming. It is capturing the meaning and the character of a concept. This capture can be performed in different forms according to Manturov, Colntzev, Sorkin, & Fedin, (1965): (1) *genetically*, by describing how the concept is created, (2) *hierarchically*, by adding new conditions to a known concept, and (3) *axiomatically*, by a set of axioms. The essence of a mathematical definition of any form is the constitution of necessary and sufficient conditions of the defined concept and the "minimality" of the set of these conditions. The variety of forms of definitions as well as the arbitrariness of the definitions (Winicki & Leikin, 2000) make this issue complicated from didactical and mathematical points of view. Different equivalent mathematical statements can serve as a definition of a specific mathematical concept. The choice of a statement is mainly based on didactical considerations. This choice is one of the critical decisions teachers make. It is a function of teachers' content pedagogical knowledge and it influences their flexibility when teaching and their ability to react to different pupils' conjectures and ideas in a pupil-centered classroom.

The program includes a collection of eleven activities designed for the professional development of secondary school mathematics teachers. The activities were developed over four years and each was adapted as a result of the teachers' reflective comments. In order to deepen teachers' knowledge of substance of mathematics, the mathematical concepts for the activities were chosen from various branches of secondary school mathematics and the definitions were based on different mathematical approaches. In order to develop teachers' capacity for pedagogical reasoning about mathematics, they were involved in authentic mathematical discussions and all the activities were designed using different cooperative learning methods (Leikin, 1997). Variations in mathematical content, in the mathematical and the didactical approaches to the definitions, and in the learning settings, enabled all the activities to be combined in a 28 hour program for mathematics teachers or students. It should be noted, however, that each of the activities can be used independently in different professional development courses. These various cooperative learning activities focused on the following mathematical concepts: absolute value, circle, parabola, special quadrilaterals, distance from a point to a straight line, distance as a function, inflection point, number, fibonacci sequence, and tangent line. The description of each one of the

---

[1] If two statements establish equal sets of objects examplifying a concept, then these statements are said to be *equivalent definitions* of the the concept. If two statements establish two sets of objects examplifying a concept one of which is strongly included into the other, then these statements are said to be *following definitions* of the the concept. If two statements establish two different intersecting sets of objects examplifying a concept none of which is strongly included into the other, then these statements are said to be *competing definitions* of the the concept.

[2] Defining is the process of choosing an approach, construction and formulation of a definition. Defining, in general, precedes any use of the definition.

activities presented in a booklet, (Winicki & Leikin, 1998) included: (a) the mathematical content of the activity, (b) the recommended learning setting, (c) working cards, (d) questions for the whole group discussion, and (e) references to appropriate literature sources.

Overall, the activities emphasise the importance of the notion of definition, and its centrality in the secondary mathematics curriculum. The activities connect mathematical concepts from various branches of secondary school mathematics. Different proving strategies and proof styles are integral parts of the activities. The set of activities includes different forms of definitions, as well as different representations used in the definitions of a concept. The assignments concerning the investigation of the logical relationships between different definitions are settled in different contexts. The activities encourage teachers to perform different tasks in order to:

- Examine logical relationships between given definitions of different mathematical concepts (e.g., Special quadrilaterals, see Leikin & Winicki, 2000);
- examine logical relationships between proposed definitions of the same concept (e.g., Absolute value, in Leikin & Winicki, 2000; Fibonacci sequence: Defining by means of symbolic rules, in this paper);
- build mathematical models in order to examine logical relationships between the definitions behind these models (e.g., Fibonacci sequence: Defining by modelling, see this paper);
- provide a definition of a concept, to examine the logical relationships between definitions provided by different teachers and to refine the definitions (e.g., Tangent line, see Winicki & Leikin, 2000).

The following two examples of mathematical activities for teachers involve learners in analysing different definitions of the Fibonacci sequence. They are similar in their learning settings and have contextual variations (Silver & Zawojewski, 1997). The first activity uses modelling of four different stories, whereas the second activity is based on four different symbolic representations of the sequence. Additionally, all the definitions chosen for the second activity are equivalent, whilst the mathematical models in the first activity are not equivalent: (the mathematical model of two of these situations is the classic Fibonacci sequence and the model of the two other situations fulfil the Fibonacci recursive rule only).

Mathematical Activities

Activities Focusing on the Fibonacci Sequence

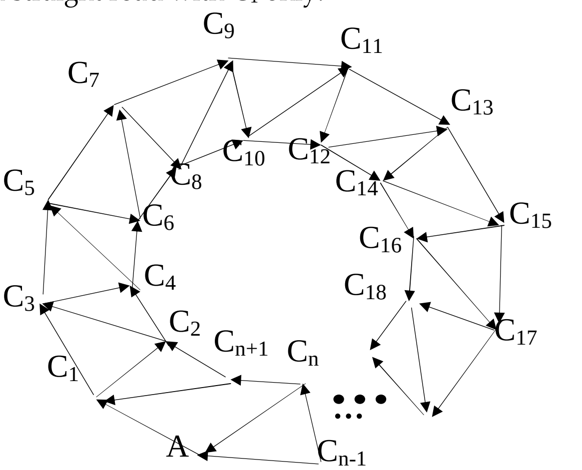
The setting. The two Fibonacci sequence activities involved the teachers in discussions on the relationships between different definitions of a concept. For each activity, each small group of teachers was presented with a working card consisting of two different definitions. Each teacher within a particular group had the same card, whilst different groups of teachers received different cards. Each definition was included on at least two cards. The teachers were asked to examine mutual relationships between the definitions on their cards and to justify their conclusions. The results of the small group discussions were presented to the whole group.

Activity 1: Defining by Modelling

Rabbits. Start with a pair of rabbits (one male and one female) born on January 1st. Rabbits begin to produce young rabbits two months after their own birth. After reaching the age of two months, each pair produces a mixed pair (one male and one female) and then another mixed pair each month thereafter. No rabbits die.

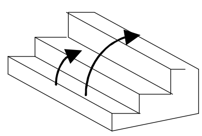
How many pairs of rabbits will there be after five months (on June 1)?
How many pairs of rabbits will there be after six months (on July 1)?
How many pairs of rabbits will there be after ten months (on November 1)?
How many pairs of rabbits will there be after *n* months?

Vectoria. In the state of Vectoria there is a road system as it is presented in the figure. All the roads in the state are one-way. The road system is closed and it is possible to get any city from each other city. A is the capital of the state. A is connected by a straight road with C_1 only.



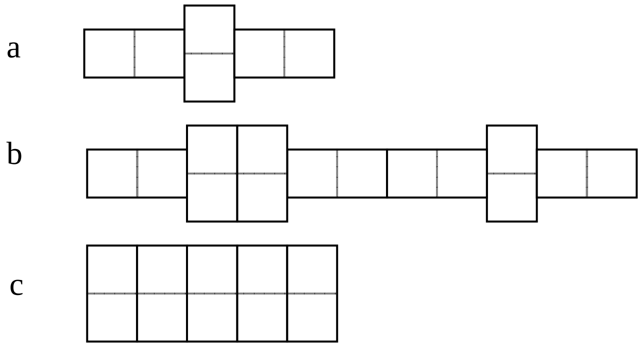
In how many ways is it possible to get from A to C_5 without visiting any C_k for $k > 5$?
In how many ways is it possible to get from A to C_6 without visiting any C_k for $k > 6$?
In how many ways is it possible to get from A to C_{10} without visiting any C_k for $k > 10$?
In how many ways is it possible to get from A to C_n without visiting any C_{n+1} and without returning to A?

Stairs. Ron climbs the stairs. He can climb one step or two steps at once. Thus he can reach the first stair in one way only. The second stair he can reach in two ways: by two single steps or one double step. The third stair he can reach in three ways: by three single steps, by one single step and then one double step, or one double and then one single step.



In how many ways can Ron reach the fifth stair?
In how many ways can Ron reach the sixth stair?
In how many ways can Ron reach the tenth stair?
In how many ways can Ron reach the n-th stair?

Dominoes. Ann uses a set of dominoes to make chains of dominoes but she ignores the value of the dots. She builds a domino chain by attaching the dominoes at the right end of the chain. In this way each position in the chain is occupied by one and only one domino, as presented in the figure. The length of chain a (see figure) is 6. The length of chain b is 11. The length of chain c is 5.



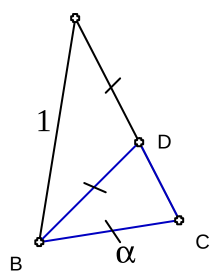
In how many ways can Ann build a chain of length 5?
In how many ways can Ann build a chain of length 6?
In how many ways can Ann build a chain of length 10?
In how many ways can Ann build a chain of length n?

Activity 2: Defining by Means of Symbolic Rules

Definition a: $$(F_n): \begin{cases} F_1 = 1 \\ F_2 = 1 \\ F_{n+1} = F_{n-1} + F_n, n \geq 2 \end{cases}$$

Definition b: $$(F_n): \begin{cases} F_1 = 1 \\ F_2 = 1 \\ F_{n+1} = \sum_{i=1}^{n-1} F_i + 1, n \geq 2 \end{cases}$$

Definition c: $$(F_n) \begin{cases} F_1 = 1 \\ F_2 = 1 \\ F_n^{\,2} - F_{n-1} \cdot F_{n+1} = (-1)^{n+1}, \ n \geq 2 \end{cases}$$



Definition d: $$(F_n): \quad F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad n \geq 1$$

α_β = –1, AB = AC

The Design

The two mathematical activities described above were designed as consecutive workshops in the framework of our program for secondary school mathematics teachers.

The first activity deals with mathematical modelling and the identification of mathematical patterns. The mathematical models for the "Rabbits" and for the "Vectoria" stories are equivalent, although less realistic than the other two situations. They lead to the construction of the Fibonacci sequence. The first situation was well known to most of the mathematics teachers, whilst the second situation was quite new. The two other situations, that is, "Dominoes" and "Stairs" are also equivalent but lead to the construction of a sequence that fulfils the Fibonacci recursive rule only and omits the first term from the Fibonacci sequence. Thus, the "Rabbits" and "Vectoria" situations described are not equivalent to the "Dominoes" and "Stairs" situations.

The notion of situations that have equivalent mathematical models is one of the focal points of the whole group discussion, as is the notion of genetic definitions. The possibility of defining mathematical concepts by providing real life situations is unusual for the teachers and thus surprising.

The other important issue of this activity is involving the teachers in the analysis of contextual variation (Silver & Zawojewski, 1997) of the situations. "Rabbits" and "Vectoria" situations may be called illustrative whilst "Stairs" and "Domino" situations are applied, as pupils in a classroom can be asked to play dominoes or to climb the stairs according to the presented rules in order to discover the pattern. The conceptual support for making mathematical sense of the questions (i.e., exploring logical relationships between different situations) is another focus of the activity.

The second mathematical activity, which focused on the Fibonacci sequence defined in a genetic way, involves a pure mathematics assignment, in contrast to the first activity. All the definitions in this activity are stated in algebraic or geometric terms. The implementation of these two activities one after another allows us to discuss with teachers the differences between pure mathematical tasks and tasks embedded in a real life context.

Definition a is the classical recursive definition of the Fibonacci sequence. Each of the three other definitions is often presented in textbooks as a property of the sequence constituting its necessary condition. During the discussion, the teachers are asked to prove necessity and to investigate sufficiency of the alternative definitions. All the statements are found to be equivalent and may be used for defining the Fibonacci sequence.

The first focal point of this activity is the criteria for choosing one of the equivalent mathematical statements as a concept definition. The equivalence of the four given mathematical statements may be proven in different ways (e.g., $a \Rightarrow b \Rightarrow c \Rightarrow d \Rightarrow a$ or $a \Leftrightarrow b \Leftrightarrow c \Leftrightarrow d$ or $a \Leftrightarrow b$, $a \Leftrightarrow c$, $a \Leftrightarrow d$). The levels of difficulty and aesthetic considerations of different proofs serve as a second focal point of the whole group discussion in this activity and support development of teachers' awareness of their own need to be flexible in their mathematical choices.

Implementation – An Example

In this part of the paper we present the implementation of one of these activities with a group of mathematics teachers, who took part in a three-year professional development course for secondary school mathematics coordinators. Ten teachers participated in the workshop "Defining by Modelling" that took place at the end of the third year of the course, which included other defining activities as well.

At the first stage of the workshop, for which half an hour was allotted, the teachers worked in small groups. Each group was given cards with two situations ("Rabbits" and "Dominoes", "Vectoria" and "Stairs", "Dominoes" and "Stairs", or "Vectoria" and "Rabbits"). After the work in small groups, the teachers presented how they approached the tasks and the main conclusions they reached. After this presentation the teachers compared the four different situations presented on the cards. Additionally, they were asked to discuss whether these situations defined a mathematical concept.

Modelling Strategies

The teachers in all the groups were eager to solve both problems they received. Thus, at the end of the time allotted, they asked for some more time to finish their work. Three groups successfully completed both tasks whilst one of the groups had difficulty modelling the "Rabbits" situation; they insisted upon accomplishing this task and therefore did not explore "Vectoria".

The teachers approached the task in different *representation modes*. For example, one of the groups who worked with "Rabbits" successfully used a *tree* representation whilst the use of a *matrix* representation constituted a performance obstacle to another group. One group of teachers working on "Dominoes", *drew* possible chains; the other group used *binary notation* for the dominoes (0 for standing and 1 for lying) and counted possible combinations of numbers. Both representations led to correct solutions.

Additionally, different *problem-solving strategies* were observed during the first stage of the work. The main strategies used by the teachers were counting all the possibilities, counting by combinatorial considerations, and working on a generic example.

Counting all the possibilities. For each one of the specific cases, n = 1, 2, 3, 4, 5 and 6 the teachers considered all the possibilities and counted them. Then, they conjectured a generalisation of the pattern for the cases of 10 and n without a formal proof. The following excerpt demonstrates such a strategy:

Roni: We just counted. To get [to] C_1 you have only one way A-C_1, the same is for C_2: A-C_2. For C_3 we have two possibilities A-C_1-C_3 and A-C_2-C_3. For C_4 we get the following: A-C_1-C_3-C_4 and A-C_2-C_3-C_4 and A-C_2-C_4, in other words three possibilities. For C_5 we saw 5 possibilities: A-C_1-C_3-C_5, A-C_2-C_3-C_5 A-C_1-C_3-C_4-C_5 and A-C_2-C_3-C_4-C_5 and A-C_2-C_4-C_5. At this moment we saw that these are all the ways of C_4 and all the ways of C_3 together (signed in color A-C_1-C_3-C_5, A-C_2-C_3-C_5 A-C_1-C_3-C_4-C_5 and A-C_2-C_3-C_4-C5 and A-C_2-C_4-C_5). In other words a_5=a_4+a_3. In the same way $a_6 = a_5+a_4$, then $a_{10} = a_9+a_8$ and we think that $a_n = a_{n-1}+a_{n-2}$.

Counting by combinatorial considerations. The teachers systematically counted all the possibilities for the requested cases of n = 5 and 6 by making combinatorial considerations. The following quotation demonstrates such a strategy:

Anat: To build a chain of dominoes of length six we can use six standing dominoes or three lying ones. The number of standing dominoes in the chain of length six must be even: two, four, or six or no standing dominoes. So the following combinations are possible: six standing, four standing one lying, two standing two lying, three lying. Now we may count all the possible permutations…

In these cases the generalisation was similar to that in the previous example.

Working on a generic example. The teachers analysed the situation in a certain case, treating it as a representative of the general one, and applied their conjecture to other concrete cases. The following excerpt illustrates this kind of teachers' work on the problem:

Tami: We first saw how we could get to the 10^{th} stair. It can happen only if I am standing on the 9^{th} stair and make a last single step or when I am on the 8^{th} stair and make a last double step. So all the cases when I get the 8^{th} stair and the 9^{th} stair are suitable... The same is with n^{th} stair we get it from $(n-1)^{th}$ and $(n-2)^{th}$ stairs, thus we saw that $a_n = a_{n-1}+a_{n-2}$

All the strategies ended up with the conclusion that each one of the four given situations might be modelled by means of a sequence of natural numbers. The discovered sequences fulfil the same recursive rule of the Fibonacci sequence and start with $a_1 = 1$, $a_2 = 1$ or $a_1 = 1$, $a_2 = 2$, depending on the situation modelled.

Defining and Proving

When all the groups had presented their results, a whole group discussion took place. This discussion focused on a set of questions that were raised partly by us and mainly by the teachers. The teachers were involved in thinking about the nature and the role of genetic definitions. We present herein some of the discussed questions and teachers' responses to these questions.

Are these situations equivalent? Firstly, the teachers discussed the equivalence of the presented situations.

Sofi: I think we had to prove that the situations were equivalent, in other words they have the same formula.

Tami: But they are not. Some of them start with 1, 1 and the two others with 1, 2.

Sofi: The rule is the same then the situations are equivalent.

Tami: The question is what kind of situations can be considered as equivalent. Is it enough for the sequences to have the same recursive rule to be equivalent, or do they have to have all the terms equal?

In this way the issue of equivalence was raised. The teachers considered dealing with logical relationship between mathematical statements usual in geometry, whereas in the case of number sequences they felt puzzled. The teachers were also surprised by the fact that real-life situations can be considered as mathematically equivalent, competing, or following, based on the logical relationships between their mathematical models.

What constitutes the model of the situations? Secondly, the teachers discussed what might constitute the model of a situation. Surprisingly, a recursive rule did not satisfy some teachers' expectations:

Ruth: What is the model? A recursive rule cannot be considered an answer to the question. It is not a perfect relationship.

Sofi: Why? Why is this recursive rule not an answer?

Ruth: You cannot find the number, which is on the 100^{th} place for the sequence immediately from the recursive connection between the terms of the sequence. That's why we did not consider it as an answer.

Sofi: The rule describes [represents] a situation and it may be a long procedure but you can find the 100^{th} number in the sequence.

Tami: I think that a_1, a_2 and the recursive rule **define** the sequence.

The need for an immediate answer to a question, and the traditional opportunity to deal with number sequences with a given general term of a sequence, led some teachers to recognise their insecurity when dealing with recursive rules. From this point of the discussion the teachers turned to the issue of defining.

What constitutes a definition? This question was the main focus of the discussion. The teachers had difficulty considering the given situations as definitions of the Fibonacci sequence. The teachers' main concern was rooted in the relationship between definition and proof.

Anat: If we proved something, is it a definition or a theorem?

Ruth: What is considered to be a definition?

Tami: Something that fulfils all the conditions and does not lead to contradictions.

Sofi: A statement, which includes necessary and sufficient conditions.

Anat: Something that makes order in our head: you know what belongs and what does not.

Tami: Are these situations definitions? What do they define?

Anat: No, it does not make order. I do not know immediately what is the mathematics rule. I have to investigate it to get to the definition.

Rachel: What difference does it make whether it is a definition or not?

Roni: The situation describes verbally a specific sequence of numbers and thus defines it.

This activity involved the teachers in a discussion of the notion of genetic definitions. Genetic definitions are those that describe the creation of a concept. In our case, it was done verbally by describing different situations. The possibility of defining by modelling was a specific characteristic of the described activity. Mathematical modelling and examination of logical relationships between different situations were followed by the need to prove that the models were indeed correct. We would like to speculate herein that a common understanding of a definition as a mathematical statement which is accepted without proof, led the teachers to a paradoxical situation in which defining was connected to proving.

Finally, the teachers agreed with Roni that "Vectoria" and the "Rabbits" situations may be considered as definitions of the Fibonacci sequence, and the two other situations define the Fibonacci recursive rule. When ending up with the identical sequences the teachers recognised equivalence of the different situations. The teachers left the workshop with an open question: *"Was this discourse mathematical or philosophical?"*

Conclusion

This paper discusses special kinds of mathematical activities for secondary school mathematics teachers focussed on the issue of mathematical definition. These activities allow teachers to be involved in discussions about the basic units of classroom mathematical discourse and thus, develop the teachers' meta-mathematical knowledge. The variety of the mathematical concepts included in the program demonstrates to the teachers the importance of the topic and its generality.

Our reflective discussions with the teachers showed that they were not always aware that there exists a possibility to define a concept in different ways and that correspondingly, the way concepts are defined determines a learning sequence, including ordering of the learning material and proving procedures.

It must be noted that this program can be used for further educational research. For example, we conducted a small investigation of the ways in which the teachers prove equivalence of different definitions of mathematical concepts as well as their attitudes towards those definitions (Leikin & Winicki, 2000), and started focusing on the relationship between defining and proving. Further research could be directed toward the following questions:

How does implementation of this program influence the teachers' classroom practice?

How does implementation of the program influence the development of teachers' mathematical knowledge?

How do teachers choose mathematical definitions and what forms of definitions do they prefer and why?
How do different forms of definitions influence the process of pupils' and teachers' learning of mathematics?
How do teachers and pupils understand the relationships between defining and proving?

References

Ball, D. (1992). Teaching mathematics for understanding: What do teachers need to know about the subject matter? In M. Kennedy (Ed.), *Teaching academic subjects diverse learners* (pp. 63-83). New York, NY: Teachers College Press.

Ball, D. (1997). What do students know? Facing challenges of distance, context, and desire in trying to hear children. In B. J. Biddle, T. L. Good & I. F. Gooodson (Eds.), *International handbook of teachers and teaching*. (pp. 769-818). Dortracht, The Netherlands: Kluwer.

Balacheff, N. (1987). Processus de preuves et situation de validation. *Educational Studies in Mathematics, 18*, 147-176.

Borasi, R. (1999). Beginning the process of rethinking mathematics instruction: A professional development program. *Journal of Mathematics Teacher Education, 2*(1), 49-78.

Chazan, D. (1993). High school geometry students' justification for their views of empirical evidence. *Educational Studies in Mathematics*, *24*, 359-387.

Cooney, T. J. (1994). Teacher education as an exercise in adaptation. In D. B. Aichele & A. F. Coxford (Eds.), *Professional development for teachers of mathematics* (pp. 9-22). Reston, VA: National Council of Teachers of Mathematics.

Comiti, C., & Ball, D. L. (1996). Preparing teachers to teach mathematics: A comparative perspective. In A. J. Bishop et al. (Eds.), *International handbook of mathematics education* (pp. 1123-1153). Dortrecht, The Netherlands: Kluwer.

De Villiers, M. (1990). The role and functions of proof in mathematics. *Pythagoras*, *24*, 17-24.

De Villiers, M. (1998). To teach definitions in geometry or to teach to define. In A. Olivier & K. Newstead (Eds.), *Proceedings of the 22nd annual conference of the International Group for the Psychology of Mathematics Education*, (vol. 2, pp. 248-255). Stellenbosch, South Africa: Program Committee.

Hanna, G. (1989). More than a formal proof. *For the Learning of Mathematics*, *9*(1), 20-25.

Kleiner, I. (1989). Evolution of the Function Concept: A Brief Survey. *The College Mathematics Journal, 20*, 282-300.

Leikin, R. (1997). A cooperative approach to learning mathematics: Guidelines and teaching materials, "Tomorrow 98" project, Haifa, Israel (In Hebrew): Department of Education in Technology and Science, Technion.

Leikin, R., & Winicki, G. (2000). On equivalent and non-equivalent definitions II. *For the Learning of Mathematics, 20*(2), 24-29.

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teacher's understanding of fundamental mathematics in China and the United States.* New Jersey, NJ: Lawrence Erlbaum Associates.

Manturov, O. V., Colntzev, U. K., Sorkin, U. I., & Fedin, N. G. (1965). *Dictionary of mathematical terms*: *Teachers' guide*. Moscow, USSR: Prosveshenie.

Sfard, A. (2000). On reform movement and the limits of mathematical discourse. *Mathematical Thinking and Learning, 2,* 157-189.

Silver, E. A., & Zawojewski, J. S. (1997). *Benchmarks of students' understanding (BOSUN) Project: technical guide*. Pittsburgh: University of Pittsburgh.

Schwartz, J. L. (1999). Can technology help make the mathematics curriculum intellectually stimulating and socially responsible? *International Journal of Computers for Mathematical Learning, 4*, 99-119.

Vinner, S. (1991). The role of definitions in the teaching and learning of mathematics. In D. O. Tall (Ed.), *Advanced mathematical thinking* (pp. 65-81). Dordrecht, The Netherlands: Kluwer.

Wilson, S. M,. & Shulman, L. S. (1987). "150 Different Ways" of knowing: Representations of knowledge in teaching. In J. Caldarhead (Ed.), *Exploring teacher thinking* (pp.104-124). Sussex, UK: Holt, Rinehart, & Winston.

Winicki, G., & Leikin, R. (1998). *Equivalent definitions: Mathematical activities for teachers and students*., Haifa, Israel (In Hebrew): Department of Education in Technology and Science, Technion.

Winicki, G., & Leikin, R. (2000). On equivalent and non-equivalent definitions I. *For the Learning of Mathematics, 20*(1), 17-21.

Authors

Roza Leikin, Faculty of Education, The University of Haifa, Mount Carmel, Haifa, 31905, Israel. Email: <rozal@construct.haifa.ac.il>

Greisy Winicki-Landman, Oranim – School of Education of the Kibbutz Movement, Kiryat Tivon, 36006, Israel. Email <greisyw@tx.technion.ac.il>