

Connecting University and Secondary Mathematics: Design Principles for Developing Prospective Teachers' Content Knowledge and Critical Stance

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In this article, we present the design of Inquiry-oriented Dynamical Systems and Modelling (IODSM), an upper-division mathematics content course that can serve as an elective for prospective secondary teachers (PSTs). The course bridges university and secondary mathematics by combining advanced content with structured opportunities to engage secondary mathematics topics, forward-looking teaching practices, and mathematical practices. Grounded in inquiry-based mathematics education and oriented toward fostering productive dispositions such as students' embracement of a critical stance, IODSM offers a learning environment where PSTs connect mathematics with future teaching. As an Illustration of Practice article, we outline a typical session, describe the theoretical grounding, and highlight four design principles that underpin classroom activities, course materials, and instructional choices. The article concludes with evidence of PSTs' willingness to critically reflect on their mathematical beliefs and values, resulting in PSTs embracing a critical stance.

Keywords • prospective teachers • university mathematics • secondary school connections • inquiry-based mathematics education

Introduction

The longstanding discontinuity between university mathematics and secondary school mathematics (Klein, 1932) remains a central challenge in mathematics teacher education. Prospective secondary teachers (PSTs) often struggle to make meaningful use of their advanced, university-level mathematical experiences in the context of secondary school teaching. A recent special issue, *Making University Mathematics Matter for Secondary Teacher Preparation* (Wasserman et al., 2023) discussed that drawing attention to how the intertwined nature of mathematics and pedagogical knowledge can be addressed through the redesign of university mathematics courses for teachers. For example, work in real analysis (e.g., Wasserman et al., 2022; Weber et al., 2020) and the Mathematical Association of America's Mathematical Education of Teachers as an Application of Undergraduate Mathematics (META Math) Project illustrated how course structures can support PSTs in developing connections between university and secondary mathematics as well as secondary teaching practices (Álvarez et al., 2024).

Alongside the focus on connecting university and secondary mathematics and teaching practices, it is equally important for mathematics teacher educators (MTEs) to consider prospective teachers' beliefs, values, and dispositions. The Association of Mathematics Teacher Educators (AMTE) Standards (2017) highlighted productive dispositions as central to teacher preparation, and such dispositions include commitments to sense-making and the willingness to critically examine one's mathematical



beliefs and practices. More broadly, the Standards called for programs to create opportunities for future teachers to develop integrated knowledge, skills, and dispositions related both to mathematics content and to the practices of teaching, which can shape and be shaped by their beliefs towards mathematical learning and teaching.

This Illustration of Practice article showcases a specific course and its curriculum design principles aimed at bridging university and secondary mathematics in ways that align with the AMTE Standards while connecting to secondary content to make the course more relevant for PSTs. The focus is on the design of an upper-division mathematics course, Inquiry-Oriented Dynamical Systems and Modelling (IODSM) that serves as an upper-division mathematics elective for prospective teachers. Grounded in Inquiry-Based Mathematics Education (IBME; Laursen & Rasmussen, 2019) with an emphasis on PSTs' embracing a critical stance (Goodchild et al., 2021), the IODSM course weaves together university-level mathematical content with structured opportunities for PSTs to engage with secondary mathematics content, forward-looking teaching practices, and mathematical practices. As this is an Illustration of Practice, this article begins with a top-level view of what a typical IODSM session looks like. We then provide details on the theoretical grounding, followed by four main sections, each of which exemplifies how the theory-informed course design is manifested in the learning environment, course materials, and instructional choices. We conclude by highlighting PSTs' willingness to critically reflect on their mathematical beliefs and values.

An Inquiry-Oriented Dynamical Systems and Modelling Classroom

A typical session in an IODSM classroom involves the instructor guiding students through a set of scaffolded tasks that begin with their intuitive and informal ideas and lead to the development of core concepts in dynamical systems with, appropriate connections to secondary mathematics content. This is accomplished through a repeating sequence of small group work followed by whole class discussion where the instructor intentionally attends to the four IBME pillars: students engage deeply with coherent and meaningful mathematical tasks; students collaboratively process mathematical ideas; instructors inquire into student thinking; and instructors foster equity in their design and facilitation choices (Laursen & Rasmussen, 2019, p. 138). Here, "students" refers to students in undergraduate mathematics classes, including PSTs. Students in the course are comprised of both PSTs and undergraduate mathematics and science majors. After the end of each session, students are asked to reply online to the prompt "Who in class, either in small group or whole class discussion, influenced your thinking and what did they say?" To begin the next session, the instructor shares three to five of the anonymised reflections, selected to highlight important ideas from the previous session, to elevate the status of student thinking in the classroom, or to highlight productive classroom interactions and behaviours, including those related to mathematical practices. The 30 students (approximately) are arranged at tables in groups of four (plus or minus one) and each group is provided with a table-sized portable white board to help facilitate small group discussion and whole class sharing. Most of the time the instructor is found moving about the room, between student groups, helping to facilitate the mathematics and promote classroom norms. Whiteboards covering the walls on two sides of the room, are also leveraged as student workspace.

Figure 1 shows a flowchart of a typical session from Unit 2: A Numerical Approach, which is representative of most class sessions. The mathematical focus of the unit is reinventing Euler's method and the connection to secondary mathematics is proportional reasoning. The purple dashed and solid arrows represent the instructor's moves during small group and whole class discussions. These moves include circulating and visiting groups in preparation for switching to whole class discussion by inviting one or more students assigned specific roles to share their group's work to the class by calling out "Spy Time" and "Gallery Walk" to foster student-to-student interaction across groups, summarising key mathematical ideas from whole class discussions, and initiating and introducing the next mathematical task. Spy Time involves the designated spy visiting one or more groups to exchange ideas and progress. A Gallery Walk involves students getting out of their seats and moving around the



classroom to look over other's work on their respective whiteboards. Whole class discussions are comprised of a mix of the instructor's and students' contributions.

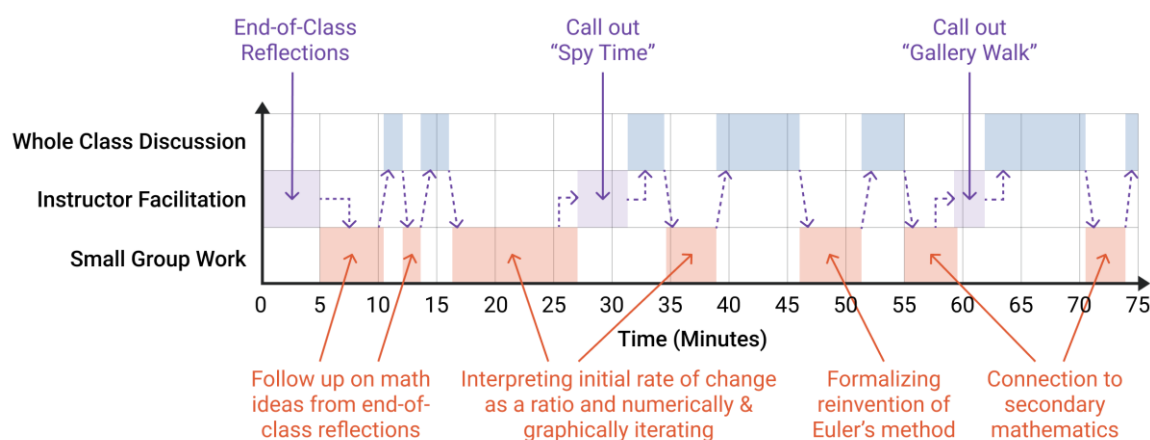


Figure 1. The time and activity breakdown of a typical session.

Notably, in an IODSM classroom, much of the intellectual responsibility lies with the students, which is reflected in the amount of time devoted to small group work. Students do much of the thinking in an inquiry-based classroom, with the instructor creating an intellectually safe space for them to do much of the intellectual work. The instructor does, of course, provide information and connect student ideas to conventional and formal mathematics (Rasmussen & Marrongelle, 2006). These contributions are largely used to consolidate and synthesise student work. Additionally, the instructor is mindful to acknowledge competency by using their name when addressing the class (e.g., “Maria’s idea is ...”) and to value different lines of thinking. In addition, the instructor seeks to create an intellectually safe space for students to change their minds and value being wrong (e.g., “How did your group revise your thinking in working on this task?”).

Furthermore, the curriculum is designed to provide PSTs opportunities, both in and outside of class, to engage in approximations of their future teaching practice. Assigned group roles (Reporter, Starter, Moderator, and Spy) allow participating students to both analyse and interpret each other student’s thinking (an important skill for teachers) as well as provide an opportunity to guide the classroom mathematical discourse through both their own small group discussion as well as whole class presentations. Finally, a variety of classroom tasks and homework problems make connections to secondary mathematics content and provide further opportunities to analyse the work secondary school students may produce.

Theoretical Grounding for the Course and Curriculum Design

This Illustration of Practice article describes an approach for connecting university and secondary mathematics, which is an evolving product of an ongoing design-based research project following the methodology described by Cobb et al. (2003). In this section, the high-level conjecture as well as the theoretical foundation that supports our overall learning goals are described. Consistent with design-based research, the theoretical perspective draws on complementary views of content knowledge and pedagogy. They are complementary in the sense that the four aspirational design principles applied view learning and teaching as inseparable from the interactive, social nature of meaning making. The high-level conjecture is:

By participating in an inquiry-oriented upper division mathematics course that embraces the four pillars of Inquiry-Based Mathematics Education, engages prospective secondary teachers in mathematical practices and forward-looking teaching practices, and revisits secondary school content related to IODSM content, prospective secondary teachers will expand their mathematical knowledge of related secondary school content and develop a critical stance regarding mathematical learning and teaching.

Figure 2 lists the four design principles and illustrates their relationship to the desired outcomes of the high-level conjecture.

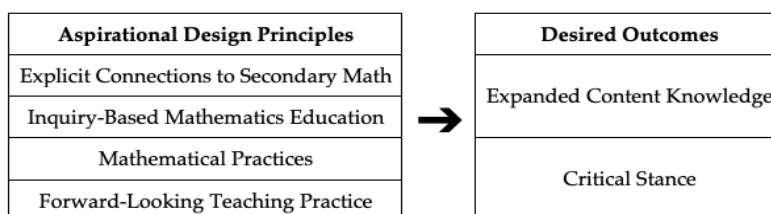


Figure 2. High-level conjecture.

We next expand on the two desired outcomes identified and then illustrate the ways in which the design principles are manifested in the IODSM course. The IODSM course provides the context for enacting these design principles. The design principles and outcomes are potentially generalisable to other upper division mathematics content courses.

Content Knowledge

Content knowledge has been described in various ways by different researchers. In this study, we draw on framings of content knowledge and pedagogical content knowledge. First, we are inspired by Skemp's (1976, 1978) construct of instrumental and relational understanding. Instrumental understanding is when one knows what needs to be done but does not have any reason or backing for why it is done, whereas relational understanding is when one considers both knowing what to do and why it is done. Knowing what to do and why to do it is critically important for PSTs. Second, to address PSTs' learning, mathematical knowledge for teaching also plays an important role in the theoretical foundations of the course. Shulman (1986, 1987) provided one of the initial delineations between content knowledge and pedagogical content knowledge. Similar to instrumental and relational understanding, Shulman argued that a "teacher need not only understand that something is so; the teacher must further understand why it is so" (1986, p. 9). Drawing on the work of Shulman, Ball and colleagues (2008) elaborated on, what they called, domains of mathematical knowledge for teaching: subject matter knowledge and pedagogical content knowledge.

The differences between content knowledge and pedagogical content knowledge are often described theoretically, however, researchers have also investigated the differences empirically. Krauss and colleagues (2008), for example, found that the degree of cognitive connectedness between pedagogical content knowledge and content knowledge in secondary mathematics teachers is a function of the degree of mathematical expertise. They argued that the development of content knowledge supports the development of pedagogical content knowledge. It follows that preparation programs for prospective teachers should focus on the importance of building relational (in the sense of Skemp, 1976, 1978) content knowledge (Baumert et al., 2010). The course is aligned with this conception of content knowledge because it is an important precursor to pedagogical content knowledge (Shulman, 1986).

Critical Stance

Curzon-Hobson (2003) defined a critical stance as "an attitude or disposition towards oneself, others and the object of inquiry that challenges and impels learners to reflect, understand and act in the milieu

of potentiality” (p. 201). We adapt this definition of critical stance to the context of the preparation of prospective secondary school mathematics teachers (Goodchild et al., 2021). This adaptation consists of three key features: awareness of experience, evaluation, and agency. PSTs must be aware of the possibility that their mathematical learning experience could be different. Consequently, they need to experience variety in the way mathematical learning experiences are designed and implemented by the instructor. Following this new experience, PSTs must be able to reach some judgment, based on their reflection and evaluation of their experience so that they might recognise when one experience of learning or teaching may be better, deeper, or richer than another. Finally, PSTs need to know their agency to change the status quo (for the better) through their own actions and future instructional practice as a secondary school mathematics teacher.

Relationships Among Desired Outcomes

We view these desired outcomes as mutually supportive. The fact that one has a different learning experience, one that highlights relational understanding, is a necessary condition for PSTs to reflect on and evaluate that experience. Consequently, the nature of extending one’s content knowledge and relational understanding is nurtured by embracing a critical stance. It is not uncommon for students, both at the secondary school level and at university, to experience mathematical learning environments that highlight or privilege instrumental understanding (Stigler & Hiebert, 1999; Stovner & Klette, 2022; Tallman et al., 2016). One of the desired outcomes of the course is that PSTs come to value and appreciate relational understanding and teaching approaches that support the development of such understanding. Developing a critical stance is a means by which this can occur. That is, by having a different learning experience, one that centres on relational understanding, and reflecting on this experience, PSTs can come to see this way of learning as better, deeper, or richer than another. This process helps inform their teaching perspective (their agency) on whether they would want to provide a similar learning experience that was meaningful to them for their future secondary school students.

Guiding Principles for Connecting to Secondary Mathematics

In this section we elaborate on the four design principles of our high-level conjecture. In each section we draw on specific examples from the IODSM curriculum, instructor moves, and/or examples of students’ experiences. These examples of students’ experiences come from the first complete iteration of IODSM at two different universities.

Design Principle 1: Explicit Connections to Secondary Mathematics

The IODSM curriculum is designed for use as an upper-division mathematics elective for PSTs focused on dynamical systems and modelling. In addition to the dynamical systems content, the curriculum includes content connections to secondary school mathematics, which begins to address the well-documented disconnect between university mathematics and secondary school mathematics. The course is referred to as Inquiry Oriented Dynamical Systems and Modelling because of the strong influence of Inquiry-Based Mathematics Education where students learn mathematics by engaging in a coherent sequence of challenging tasks (Laursen & Rasmussen, 2019).

In contrast to a typical differential equations course with an analytic technique heavy approach, IODSM embraces a range of graphical, numerical, and qualitative approaches for analysing how quantities evolve over time based on models of their rate of change. As an example, the phrase “rate of change equation” is often used in place of “differential equation” throughout the materials to emphasise the dynamic nature of how quantities change. Furthermore, most units in the curriculum begin with real-world contexts and engage students in varying levels of developing rate of change equations to model the phenomenon of interest.

As our brief review of the literature revealed, PSTs do not typically see how their study of university mathematics relates to the content they will teach in secondary school. To make these



connections explicit, the in-class course materials contain one or more call out boxes spread throughout the IODSM tasks. Facilitated by the instructor, these call out boxes provide an opportunity for students to make connections to secondary school mathematics while in the midst of learning IODSM content. Blue call out boxes refer to specific secondary school content, red call out boxes refer to specific mathematical practices, and green call out boxes refer to the recurring themes of modelling. Each unit ends with a section that explicitly develops a specific content connection to secondary school mathematics. Figure 3 provides examples of various call out boxes that appear throughout the curriculum.



Figure 3. Examples of call out boxes from various units.

Each unit also contains a set of homework problems. The first section of every homework set begins with several problems under the header “Connections to Secondary Mathematics.” We chose to lead with homework problems that make further connections to secondary school to highlight the centrality of this part of the curriculum. This is followed by a second section with 10 or more problems under the header “IODSM Content.” To give the reader a more complete sense of the structure for each unit, Unit 2 in its entirety is included in Appendix 1. There is a total of nine units structured similarly. Table 1 shows the IODSM content of each unit and the specific connection made to secondary school mathematics. Enquiries regarding the IODSM course materials may be directed to the second author, whose contact details are provided in the acknowledgements.

Table 1
IODSM Units and Secondary Mathematics Connections

Unit	Secondary Mathematics Connection
1: Qualitative and Graphical Approaches	Functions
2: A Numerical Approach (Euler’s Method)	Proportional Reasoning
3: An Analytic Approach (Separation of Variables)	Function Translation
4: Linear Differential Equations	Modelling
5: Autonomous Differential Equations	Make Use of Structure
6: Modelling with Autonomous Differential Equations	Covariational Reasoning
7: Effects of Varying a Parameter in Autonomous Differential Equations	Polynomial Roots
8: Introduction to Systems	Parametric Equations
9: Spring Mass System and Linear Systems	A Retrospect

To further illustrate the nature of the secondary mathematics connections, we showcase the secondary mathematics connections (both end-of-unit in class activities and the homework) for Unit 3. The IODSM topics covered in Unit 3 include comparing approximate and exact solutions, the analytic solution to separable differential equations, and the structure of graphs of solution functions. Before highlighting the secondary connections, we briefly review the IODSM content that sets up the secondary school mathematics connection.

In Unit 3, students develop the technique of separation of variables to derive the general solution for separable differential equations and then explore the relationship between the slope fields for an autonomous rate of change equation, a time-dependent differential equation, graphs of their respective solution functions for different initial conditions, and their respective general solutions. Autonomous differential equations are ones where the rate of change of the dependent variable depends only on the value of the dependent variable itself. The symbolic form of an autonomous differential equation is $dy/dt = f(y)$. Graphs of non-equilibrium solution functions to autonomous rate of change equations are horizontal shifts of each other and the general solution manifests this structure.

The general solution to autonomous differential equations contains the term $(t + c)$, which parallels the quadratic equation $g(x) = (x + c)^2$. Differential equations where the rate of change in the dependent quantity only depends explicitly on time have the form $dy/dt = f(t)$ and the general solution contains the term “+ c”, which parallels the quadratic function $h(x) = x^2 + c$. In the IODSM context, students explored how solution function graphs to two different rate of change equations were, in one case, horizontal shifts of each other and in the other case vertical shifts of each other.

This IODSM content laid the groundwork for the in-class end-of-unit section on connections to secondary mathematics, which as noted in Table 1 is “Translations.” In this section students are first asked to explain why the graph of $f(x) = x^2 + 5$ is a vertical shift up five units of the graph of $g(x) = x^2$. Similarly, they are asked to explain why the graph of $f(x) = (x + 3)^2$ is a horizontal shift three units to the left of the graph of $g(x) = x^2$. The end of the unit section on connections to secondary



mathematics contained two additional problems that further developed students' relational understanding (Skemp, 1976) for the horizontal translation as well as an open-ended card sorting task containing several different parent functions and related functions.

For homework, students had two follow-up problems that focus directly on the horizontal translation relationship. In the first problem, students were asked to use the composition formation, $g(x) = f(x + 2)$, for functions $f(x) = x^2$ and $g(x) = (x + 2)^2$ to explain why the graph of g is shifted two units to the left of the graph of f . In the second problem, students were asked to read the article "*Conceptions of function translation: Obstacles, intuitions, and rerouting*" by Zazkis et al. (2003), interview a student (college or secondary school) or classroom teacher using the same protocol as in the paper, and then write a summary of the person's reasoning with appropriate connections to the reading.

Unit 3's connection to secondary school mathematics addresses the need for PSTs to have an understanding that goes beyond a rule-based explanation for why one graph is a horizontal shift and in what direction to its parent function. Research has consistently shown that horizontal translations of functions are conceptually challenging and counterintuitive (e.g., Yim & Lee, 2021; Zazkis et al., 2003). Such relational understanding positions PSTs to be able to teach their future secondary school students this transformation and related function transformations with meaning. The multiple explanations for why one function is a horizontal shift in a particular direction, the article, and the interview also addresses PSTs pedagogical content knowledge. Several other units also include both content and pedagogical content knowledge connections.

Design Principle 2: Inquiry-Based Mathematics Education

IBME is a theoretical framework for instructional practices that is comprised of four pillars (Laursen & Rasmussen, 2019, p. 138), which are:

1. students engage deeply with coherent and meaningful mathematical tasks;
2. students collaboratively process mathematical ideas;
3. instructors inquire into student thinking; and
4. instructors foster equity in their design and facilitation choices.

The first two pillars focus on what students do and the second two pillars focus on what the instructor does. All four pillars are aspirational in the sense that they highlight expectations but do not prescribe how a classroom community might meet these aspirations. Regarding student activity, the first pillar stresses how deep engagement occurs when students dive into the important ideas of the mathematical content over time. The second pillar aspires for students to collaboratively process ideas, because as students interact and problem solve together, it deepens understanding, builds communication, and shows the many different ways of thinking that lead to solutions. The last two pillars emphasise instructor moves. Pillar 3 states that instructors inquire into student thinking. This means that an instructor is genuinely interested in how students reason and strive to interpret rather than evaluate this reasoning. Importantly, the intention is that the instructor not only values student thinking but uses their thinking and reasoning to advance the mathematical agenda (Kuster et al., 2018; Plaxco & Wawro, 2022; Rasmussen & Marrongelle, 2006). The fourth pillar describes the importance of fostering equitable instructional practices. Here, instructors must be aware of not just student thinking, but of their experience and feelings within the classroom, ensuring students are listened to, acknowledged, and respected.

To expand on the first two pillars, the design of the IODSM curriculum is inspired by the instructional design theory of Realistic Mathematics Education (RME; Freudenthal, 1973). RME views mathematical concepts, structures, and ideas as inventions that humans create to organise the phenomena of the physical, social, and mental world. A central heuristic is *guided reinvention*. The term reinvention is used to signal that students are building on their informal and intuitive knowledge to lay the foundation for conventional and formal mathematics. Students' own reasoning and intuitive or informal ideas are used by the instructor to guide this reinvention process (Gravemeijer & Doorman,



1999; Yackel et al., 2003). This aligns with the challenging and meaningful tasks described in IBME Pillar 1 and instructor leveraging student thinking in IBME Pillar 3.

We illustrate the first two pillars with course materials in Unit 2: A Numerical Approach. Class problems in Unit 2 scaffold the process of developing a conceptual foundation for reinventing Euler's method. The problems are sequenced so that students first approximate graphical solution functions by informally "stitching together" tangent vectors on the graph. Experimenting by hand and with a GeoGebra applet supports students' graphical reinvention of Euler's method, which then leads to a table of values reflecting the graphical tip-to-tail approach numerically. Through this process, students construct their own model of Euler's method, experimenting with ideas that lay the conceptual foundation for Euler's method. The unit ends with students creating their own symbolic expression for their graphical and numerical approach. The instructor formalises and validates the reinvention by introducing Euler's method with the "Now-Next" language and notation as a way of connecting recursive reasoning with formal sequence notation (Lannin, 2003). This way, students develop the algorithm for Euler's method that formalises their intuitive, graphical, and numerical approaches with the conventional algorithmic procedure. The RME curriculum design as illustrated in Unit 2 was to support students' engagement in meaningful and coherent mathematical tasks (Pillar 1).

For collaborative learning (Pillar 2), students actively participate in small group and whole class discussions during typical class sessions, as shown in Figure 1. As mentioned, there are different group roles, one of which is the Reporter. The Reporter's job is to share back to the whole class about their group's progress and solution, however tentative. In the course design, there are three other group roles: Starter, Moderator, and Spy. The Starter reviews the task, requests clarifying questions, and seeks to establish initial common ground among small group members. The Moderator makes sure everyone in the group gets a chance to talk, to have their ideas heard, and that the group helps members learn from one another. The Spy visits other groups to learn about their progress, ask questions, and exchanges insights about their group's work.

Group roles help structure and foster collaboration in small groups by ensuring that intellectual responsibilities are shared. In addition, students reflect on their peers' end-of-class reflections and engage in peer review of homework assignments, which exposes them to different ways of thinking and reasoning. These activities help elevate collaborative work and encourage students to value others' perspectives. In fact, many students report that these opportunities for collaboration broaden their perspectives and strengthen their own understanding. For example, in end-of-class reflections from Unit 2, an IODSM student shared that:

What influenced me the most was the way we were collaborating and double-checking each other's work. This group work had a positive impact on myself and made me want to solve more problems like this and continue our collaboration.

This student highlighted how collaboration not only motivated them to pursue solving more problems but also provided peer validation and accountability within the guided inquiry process. Similarly, another student mentioned how their understanding of the definition of rate of change has changed and been enhanced by working with other students:

Talking about the definition of rate of change, and how our definitions have changed through the past three weeks has influenced my thinking. Several groups mentioned how they see rate of change as a function instead of just the derivative, which is a perspective I never saw, despite treating it as a function for the last few classes.

The previous reflection illustrates how collaborative learning can reshape one's own mathematical understanding by reviewing others' perspectives, which one might not encounter when working alone. These two examples help support the intention that the norms related to collaborative learning established in the course yield both mathematical and intrinsic benefits for students.

Collaboration is also built into assessments. During exams, the instructor allows approximately 15-minutes for students to discuss problems in small groups without writing anything down, midway through a 90-minute exam. This practice values mathematical collaboration while still assessing



individual understanding. Connecting Pillar 2 back to Pillar 1, students' mathematical reinventions do not occur in isolation; they were supported and deepened through collaborative learning structures that shape how students engage with the content.

Regarding the two instructor pillars, during small group work, the instructor continually circulates around the room, prompts students to explain and articulate their mathematical ideas to their peers, and invites students who are assigned as Reporters to represent the group for class sharing. As briefly illustrated in Figure 1 with the purple arrows, the instructor facilitation involves recording key mathematical ideas that emerge across groups and sharing them during whole class discussion by asking students to comment on and interpret different approaches. Importantly, credit is explicitly given to individual students or groups, emphasising that the advancement of the mathematical agenda arises from students' contributions. This is how Pillar 3, instructors inquire into student thinking, is operationalised in IODSM.

Students' end-of-class reflections are also leveraged in a different way here than in Pillar 2. While they present multiple ways of thinking for collaborative learning (Pillar 2), reflections also serve as a routine for reviewing and consolidating ideas or as a springboard for the next lesson segment, surfacing students' thinking (Pillar 3). Additional inclusive instructional practices include rotating group roles to foster equitable participation, thoughtfully composing small groups with at least two women in a group of three or four, and ensuring that each student's thinking is represented at some point in class through anonymous reflections. Practices such as learning students' names and having students create a personal introduction slide create a comfortable intellectual community where every student is valued. These instructional practices enact Pillar 4; instructors foster equity in their design and facilitation choices.

Taken together, the four pillars guide how the IODSM course is intentionally designed to intertwine meaningful mathematical tasks, collaborative learning, inquiry into students' thinking, and equitable instructional practices to provide meaningful and influential learning experiences to PSTs. Pillar 1 focuses on how students reinvent mathematical concepts, such as Euler's method, through solving carefully sequenced tasks as well as having explicit connections to secondary mathematics in order to obtain intellectual ownership and expand content knowledge. Pillar 2 shows how collaborative learning, through group roles, discussions, and reflections, enables students to build understanding collectively and value diverse perspectives. Pillar 3 emphasises the instructor's role in curating and amplifying student thinking, positioning students' contributions at the centre for whole class discussions and mathematical discourses. Pillar 4 underscores an eye towards equity in instructional practices, including rotating group roles, attending to group composition, and leveraging anonymous reflections that ensure all students' voices are represented and recognised.

Design Principle 3: Mathematical Practices

Mathematical Practices can be seen as proficiencies, competencies, or process goals, and many countries incorporate the practices in mathematics curricula. For example, in Denmark the KOM framework (Competencies and the Learning of Mathematics) (Niss, 2003), specifies eight mathematical competencies (e.g., thinking, problem tackling, modelling, reasoning, representation, symbols & formalism, communication, aids & tools). The *Australian Curriculum: Mathematics* (2026) includes explicitly four proficiency strands: Understanding, Fluency, Problem Solving, Reasoning—described as the “actions” students engage in while learning the content. In the United States, the *Common Core State Standards* (2010) delineates eight mathematical practices (MP1–MP8). These practices are ways in which mathematics educators can engage students at all levels in different ways of doing and learning mathematics in the classroom.

These practices are ones that both K–12 educators aim to enact with their school students as well as MTEs aim to enact with their PSTs. Research indicates, however, that it is sometimes difficult for PSTs or in-service teachers to understand the nuances of the standards (Bleiler et al., 2015). Therefore, the units were designed intentionally for these practices to emerge, often overlappingly, with



opportunities for PSTs to reflect explicitly on their engagement in the standards. The eight mathematical practices, which are described in more detail in Appendix 2 are:

- MP1: Makes sense of problems and persevere in solving them
- MP2: Reason abstractly and quantitatively
- MP3: Construct viable arguments and critiquing the reasoning of others
- MP4: Model with mathematics
- MP5: Use appropriate tools strategically
- MP6: Attend to precision
- MP7: Look for and make use of structure
- MP8: Look for and express regularity in repeated reasoning

The mathematical practices design approach in the IODSM course incorporates purposefully several opportunities in class for students to reflect explicitly on the ways in which their IODSM work embraces one or more the mathematical practices. These opportunities are supported by the red call out boxes included in the course materials as well as occasions when the instructor makes in the moment references to specific mathematical practices. In addition, students complete a homework assignment that explicitly requires them to reflect on their engagement with the eight mathematical practices throughout the course. In this assignment, students (a) revisit the description of each mathematical practice, (b) write examples from their classwork that illustrates how they engaged in each practice, (c) compare how their experience with mathematical practices in the course was similar or different from other mathematics courses they have taken, and (d) state aspirations for their future teaching in relation to the mathematical practices. The assignment, which is given near the end of semester, provides a structured opportunity for PSTs to connect their learning with their future instructional practices.

Students' engagement with the mathematical practices is designed into course problems across every unit and every class session. For example, in Unit 1, the intention is that students directly engage with MP2 as they compare algebraic and differential equations and justify whether proposed functions were valid solutions. Unit 2 emphasises MP1 through numerical approximations and recursive algorithms, where students refine their approximations and eventually reinvent Euler's method. Also, there are explicit connections to MP4 in Unit 2 when students move from an unrealistic unlimited growth model to one that fits the real-world scenario and MP8 by writing down Euler's method and recursive algorithms. Unit 4 directly connects with MP4, in particular the Salty Tank problem, where students build and solve equations to represent inflows and outflows of salt water in a tank. Students also have the opportunity to engage in MP7 by investigating the reverse product rule and using it to find general solutions to linear differential equations. Unit 7 highlights MP3 in the fish harvesting problem, where students propose, critique, and revise differential equation models with parameters. MP7 is also evident in this unit with the use of bifurcation diagrams and phase lines which require students to track how equilibrium solutions change when parameter varies.

One particularly strong point of overlap between the course design and IBME Pillar 2 is the emphasis on MP3. In IODSM, group work and whole-class discussions are structured so that students make conjectures, justify their thinking, and publicly engage with peers' reasoning. For example, when students present different approaches to solving problems, they are expected to not only defend their own method but also critique others, and re-shape their own if necessary. This aligns directly with IBME Pillar 2 in how, in collaborative mathematical activity, intellectual ownership is shared and knowledge is co-constructed among students.

We conclude this section by taking a deeper look at Unit 5: Autonomous Differential Equations, which provides an explicit opportunity to engage in MP7, look for and make use of structure. Unit 5 begins with a population modelling context with limited resources (referred to as the logistic model). Students are given the rate of change equation $dP/dt = 0.3P(1 - P/10)$ and invited to sketch, based solely on the sign and magnitude of the rate of change equation, qualitatively correct graphs of solution functions. Students are then prompted to reflect on the minimum number of non-equilibrium solutions needed to represent all solution functions. This task harkens back to their work in a previous unit where

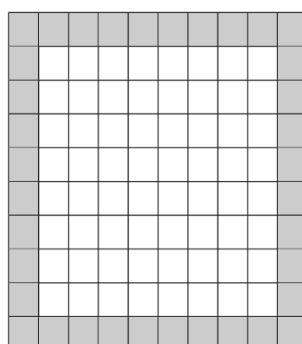


they figure out that graphs of solutions to autonomous rate of change equations are horizontal translations along the t -axis. As such, there is an inherent structure of the space of solution functions where in between equilibrium solutions (which in this case are $P(t) = 0$ and $P(t) = 10$), only *one* graph is needed to represent the entire solution space. Classifying the equilibrium solutions as either an attractor or a repeller provides further structure. This sequence of tasks is then repeated for a modification to the logistic model with the additional task of inviting students to create a one-dimensional representation of the two-dimensional space of solution function graphs, essentially reinventing a phase line. In the final IODSM task for Unit 5, which has header “Seeing Structure in Graphs of Solutions to Autonomous Differential Equations,” students are given the context free differential equations $dy/dt = y^2(y - 3)$ and asked to sketch representative solution graphs and the corresponding phase line, classify equilibrium solutions, and to explain how a person could “shift their perspective to see that the phase line is a one-dimensional projection of all of the two-dimensional $y(t)$ graphs of solutions.”

Like all other units, Unit 5 ends with an explicit connection to secondary mathematics. The header for Unit 5 connection is “Make Use of Structure: Connections to Secondary Mathematics” followed by a red call out box detailing MP7. The first task in the secondary connections is to reflect back on their work to date and identify instances when they either applied general mathematical rules to specific situations, looked for the overall structure and patterns in mathematics, or saw complicated things as single objects or as being composed of several objects (all three of which are key points in MP7). The way in which students see structure in the solution space to autonomous rate of change equations exemplifies seeing complicated things as being composed of several objects.

The secondary connections portion of Unit 5 then transitions to a secondary school task where, as stated in the prompt, “looking for and making use of structure is an integral component of the problem.” The task that students work on, referred to as the Border Problem, is adapted from the work of Boaler and Humphreys (2005). As shown in Figure 4, students are given a 10×10 grid where all the border cells are shaded, and students are asked to find multiple ways to figure out the number of shaded squares without counting them one by one.

8. Let’s do a problem that can be done in secondary school where looking for and making use of structure is an integral component of the problem. Below is a 10×10 grid with its “border” shaded.



- (a) Without counting one by one, figure out different ways to determine the number of shaded squares and explain your method for each way.

Figure 4. The Border Problem (Boaler & Humphreys, 2005, p. 15).

One student strategy for seeing structuring in the number of shaded cells is to see the top and bottom row as containing 10 shaded cells each with the two side rows each containing eight shaded cells, which can be recorded by the instructor as $10 + 10 + 8 + 8$. Another strategy is to see the corners as four individual shaded cells which leaves the two columns and two rows as each containing eight

shaded cells, which can be recorded as $4 + 8 + 8 + 8 + 8$. These are just two of the many ways that students can and do see structure in the 10×10 grid. After creatively finding many different ways to determine the number of shaded cells, the further question asks students, “Which of the different ways to determine the number of shaded squares required from you the greatest ‘shift in perspective’?” which harkens back to the MP7 activity of seeing complicated things as being composed of several objects.

In the homework, IODSM students go even further with this problem by watching a video of Ms. Humphreys teaching the Border Problem with her middle school students. After watching the video, where often the middle school students had similar strategies to the IODSM students in class, IODSM students were then prompted to reflect on several questions. In addition to touching on mathematical practices, these questions invite PSTs to reflect critically on the mathematical opportunities and the teacher moves, and to imagine themselves teaching (or not) in a similar way. The latter of which relates strongly to practices detailed in the next section on Design Principle 4: Forward-looking Teaching Practices.

Design Principle 4: Forward-looking Teaching Practices

Forward-looking teaching practices include approximations of professional teaching practices and opportunities to reflect on one’s future teaching. Llinares (2013) defined professional teaching practices as consisting of the following three components: (1) interpreting and analysing students’ thinking; (2) initiating and guiding the mathematical discourse and interactions in the classroom; (3) selecting and designing suitable mathematical tasks. The first two professional practices are enacted in IODSM as forward-looking teaching practices. As there were some non-PSTs in IODSM, we sometimes generalise these to forward-looking professional practices. As IODSM is a mathematics content-focused course, we do not have PSTs engage in the third approximation of professional practice. Selecting and designing mathematical tasks is often done in methods courses rather than content mathematics courses.

The first forward-looking teaching practice, interpreting and analysing students’ thinking, is achieved in three ways. First, our future teachers regularly interpret and analyse each other’s thinking during small group work and whole class discussions. While the professional practice from Llinares refers to analysis of school students thinking, our IODSM students are engaged in interpreting and analysing the thinking of their peers. End-of-class reflections reflect this forward-looking teaching practice by asking PSTs to notice and make sense of others’ reasoning; recall that the daily prompt—“Who, either in your small group or in the whole class discussion, influenced your thinking today and what did they say? Please provide a little bit of detail,”—requires students to interpret and makes sense of someone else’s thinking.

Peer review of homework is a second way in which students are positioned to interpret and analyse mathematical arguments and explanations generated by others. In giving feedback, they must interpret their peers’ reasoning, anticipate potential misunderstandings, and decide what kind of questioning or prompting might clarify their peers’ work. Providing feedback also gives students practice in initiating and guiding mathematical discourse, since they learn to press their peers toward clearer reasoning and stronger arguments. These interactions mirror the kinds of discourse moves teachers make in their classrooms.

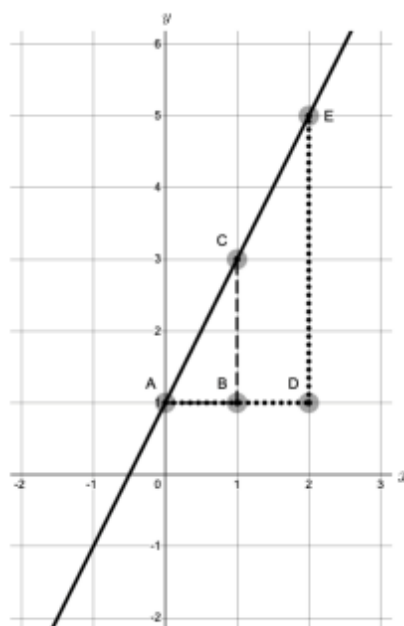
A third way the first forward-looking teaching practice is achieved is through in-class and homework problems that highlight common challenges faced by secondary school students, which prompt PSTs to examine carefully student thinking. For example, a Unit 2 problem, which connects to reinventing Euler’s method where one holds the rate of change constant over a set time interval describes how a hypothetical student, Max, misunderstood slope as a constant rate of change (see Figure 5). Max drew two slope triangles on the same line but concluded that the larger triangle represented a larger slope. PSTs analyse Max’s reasoning, explore the source of his misconception, and consider how they would respond instructionally. This problem encourages PSTs to envision future



teaching decisions and how to support effectively school students in recognising the meaning of a constant slope.

8. In the tip to tail Euler's method, one uses a constant rate of change over a specific time increment. Graphically this is shown as a constant slope. Conceptually understanding constant slope can be challenging for secondary school students. This next problem unpacks one of these challenges.

Max is trying to determine the slope of the line graphed below. He drew in the dashed slope triangle ($\triangle ABC$) and dotted slope triangle ($\triangle ADE$) and concluded that the slope of the dotted triangle is larger than the slope of the dashed triangle because the dotted triangle has a vertical change of 4 units while the dashed triangle only has a change of 2 units.



- (a) Develop an explanation that would help Max understand that the dotted and dashed triangles represent the same constant slope.

Figure 5. Unit 2: Constant rate of change problem.

The second forward-looking teaching practice, initiating and guiding the mathematical discourse and interactions in the classroom, is achieved through the use of group roles. Group roles rotate every class and are designed to support productive student interaction and equitable participation. Each group role is intended to also embody the approximation of teaching roles that initiate and guide the mathematical discourse and interactions. The Starter initiates their small group work by making sure everyone understands what the problem is asking, clarifies terminology, and prompts mathematical conversation to problem-solving. The Moderator pays close attention to group dynamics and attends to everyone's opportunity to contribute to the group discussion and progress. The Spy participates in different groups' conversations, interpreting other students' work as well as having to engage in others' thinking to answer their questions by comparing group work, and the Reporter summarises the mathematical discourse that took place and presents it to the class, similar to how an instructor might. In sum, all four group roles act as an approximation of the act of teaching as they interpret students' thinking and present mathematical ideas in front of the class but are situated within the context of the PSTs' own learning processes which approximate teaching practices.

Finally, the IODSM approach complements the two forward-looking teaching practices with two autobiographies, the second of which invites PSTs to reflect on their future teaching, which is the third

component of a critical stance (agency). The first autobiography, which is assigned at the beginning of the semester and sets the stage for the second autobiography, asks them to reflect on their prior learning experiences in secondary and college mathematics, describe how they identify themselves mathematically (including beliefs and values towards mathematical learning and teaching), and anticipate the kind of classroom they hope to create once they become a teacher.

At the end of the semester, PSTs write a Mathematical Teaching Autobiography that draws on both their initial autobiographies and their IODSM learning experiences. This second autobiography pushes PSTs to imagine their future professional selves, examine their evolving beliefs about teaching and learning mathematics, and compare how they identified mathematically in the past versus the present. As learners and future teachers, PSTs typically compare their past mathematical learning experiences with their IODSM experiences, reflect on their engagement with opportunities such as enacting group roles, responding to hypothetical secondary students' thinking, connecting university mathematics to secondary mathematics (Design Principle 1), and practicing mathematical practices (Design Principle 3) that approximate professional teaching practices and mathematical skills. In doing so, PSTs articulated explicit aspirations for the teaching practices they hope to carry forward and whether or not they would adopt inquiry-oriented teaching practices in their future classrooms. The enactment of Design Principle 4, therefore, is one way in which MTEs can support PSTs adoption of a critical stance where they evaluate their prior beliefs about the nature of mathematics and feel empowered to shape their future classrooms to impact positively on secondary school students' learning and classroom experiences.

Embracing a Critical Stance

A critical stance comprises three key features: awareness of having a new experience; evaluation of this experience as better, deeper, or richer than a previous experience; and agency to improve the status quo for their future students. Preliminary analysis of the most recent iteration of the IODSM course indicates that PSTs did embrace a critical stance. For example, one PST in their second autobiography outlined their changed perspective on learning and mathematics identity at the end of their IODSM course. Throughout the quotation, connections to one or more of the design principles are inserted in brackets.

I find now that the parts of mathematics that I claimed to love the most in my first biography were almost entirely excluded from our class, yet I had more fun in this class than almost any other previous math course I might have thought. I liked more procedural learning of mathematics, but I think that I thought that only because I was good at it and it made me really successful in my education. But this class, and classes I've taken like it, have been very different, using inquiry-oriented techniques, focusing on group work and group discussion [Design Principle 2], and those classes I've enjoyed the most and they've made the biggest impact on me. A fundamental part of my development as a mathematician through these classes has been learning about the Standards for Mathematical Practice [Design Principle 3]. These practices put into words what math should look like in such a tangible way, and getting to experience them has given me the first-hand experience to see how important they are. It made me realise how little I got to actually do math in lecture-based classes and how that led me to favour memorisation over engaging in the material. Now I see my math identity as being an active mathematician who can engage in material, persevere to solve problems, be creative and inventive, work with others to come up with solutions and both reach understanding of material. All of the Standards for Mathematical Practice are a huge part of my mathematical identity now.

This PST expressed appreciation for the IBME pillars in Design Principle 2 and the mathematical practices in Design Principle 3 to be two of the features that were influential to their learning. They also embraced a critical stance as they were aware how inquiry-oriented learning was different from their prior experiences. By evaluating this different learning experience from IODSM, they found more reasons to enjoy learning mathematics in relation to the design principles.

Another PST not only discussed their enjoyment of an inquiry-oriented style of learning but also commented that now it is "hard to enjoy traditional classroom styles again." This comment reflects



their thoughtful evaluation of prior (and potential future) mathematical learning experiences. They also commented on their growing identity as a mathematician:

I absolutely love that we were able to explore ideas such as Euler's Method and we rediscovered it by developing it ourselves through exploration and extensive reasoning. I feel like these little milestones really help me see myself as a mathematician, just like Euler!

In addition, comments were made about their agency to teach their future students with meaning:

At one point or another I will be working with [secondary] students on this topic and I am proud to say that I will explain why the functions shift rather than just stating that they do.

As a final example of PSTs embracing a critical stance, the following reflection addressed teaching identity and forward-looking teaching practices:

One of my biggest takeaways from this class is that I want to give my future students more agency in their learning of math. I used to think effective teaching was all about providing the clearest examples, concise explanations, and organising lecture notes very well for my students. To me, teaching was more about feeding students information and giving them time to practice it on their own when they did their homework. I love math and I really enjoy explaining concepts, but I've realised it's a disservice to my students if I immediately give them the steps and answers. If I dominate the problem-solving process and provide the solution too quickly, I'm robbing them of a meaningful and rewarding learning opportunity. When students come to tutoring, I now try to avoid writing out the full solution for them using my marker and whiteboard. Instead, I want to give them opportunities to explain their reasoning in their own words and write out their steps on the whiteboard.

Importantly, this PST emphasised a shift in their perspective of teaching with agency, which is that teaching mathematics does not mean merely feeding information; instead, the intellectual responsibility should lie with students. These example quotations from PSTs support our conjectured outcome of embracing a critical stance, through their reflections as learners and prospective teachers and willingness to examine their beliefs, after experiencing the learning that IODSM offered. Further evidence of PSTs in IODSM embracing a critical stance can be found in Qiu et al. (in press).

Conclusion

In this article, we detailed four design principles that support connecting university and secondary school mathematics and showcased how these principles are realised in the development of course materials, activities, and teaching strategies. We conjectured (and have a growing body of evidence for the conjecture) that these four design principles support the fulfilment of the two desired outcomes: expanded mathematical knowledge of related secondary school content and the development of a critical stance regarding mathematical learning and teaching. Previously reports on initial enactments of IODSM included discussing PSTs' expanded content knowledge, changed beliefs, and embracement of critical stance (Apkarian et al., 2023; Goodchild et al., 2021; Qiu et al., in press). The design principles that we have shared here were informed from these previous findings. We also have begun conducting empirical studies using the most recent (and complete) iterations of IODSM (e.g., Rasmussen et al., 2026; Santos et al., 2026), and there are other iterations to be implemented in the future. We encourage other MTEs to also conduct similar empirical studies.

For future work, our multi-year design-based research across institutions will continue to collect more evidence to examine the high-level conjecture across different PST populations. Another important goal of the project is to provide professional development and workshops that support instructors in adapting the IODSM course model and curriculum, or designing their version of inquiry-oriented courses deemed suitable for their PSTs' learning needs. To achieve this goal, this Illustration of Practice article leads by example by showcasing the potential of one possibility. In an IODSM PST's words:



I've realised that the curriculum for this class was carefully and intentionally designed to foster student discoveries. It's not easy to lead a fully inquiry-based class. I can imagine it's very time consuming not just to plan the lessons themselves, but also plan how to facilitate discussions, anticipate misconceptions, and manage group dynamics Thankfully, I don't have to change my entire teaching style from one day to the next. I can start by integrating inquiry-based practices in manageable ways.

As this PST pointed out, implementing an inquiry-oriented approach has its challenges. More importantly, this PST reminds us that one does not need to create or "lead a fully" redesigned course; MTEs can start by integrating more opportunities to prepare PSTs "in manageable ways." The IODSM course illustrates one such effort, connecting university mathematics with secondary school mathematics, forward-looking teaching practices, and mathematical practices, in ways that support PSTs in developing both deeper content knowledge and productive dispositions, including the embracement of a critical stance.

We offer this Illustration of Practice through the IODSM course not as a template to be replicated in full, but as a model from which MTEs may be inspired to adapt elements and aspects of the four design principles that best suit their teaching contexts. We look forward to seeing creative extensions and new approaches that continue to address the longstanding challenge of connecting university mathematics with secondary mathematics.

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Competing interests

The authors declare there are no competing interests.



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Appendix 1: IODSM Unit 2



Unit 2: A Numerical Approach

A Rate of Change Equation for Limited Resources

STANDARDS FOR MATHEMATICAL PRACTICE

A key feature of **Mathematical Practice 4: Model with Mathematics** is making adjustments by recognizing when a model might need to be refined or adapted to better fit the real-world scenario. Let's explore this!

While the unlimited growth model $\frac{dP}{dt} = 0.3P$ (or more generally, $\frac{dP}{dt} = kP$) is useful in some contexts, in most population contexts unlimited growth is not realistic because resources (space, food, etc.) are limited. So, we need to modify the unlimited growth model to better fit real-world scenarios.

1. Based on real world data, scientists have found that the following three features accurately model many different population growth situations where the resources are limited:
 - (i) when the population is very small, the rate of change is similar to unlimited growth,
 - (ii) as the population grows, the rate of change becomes less positive, and
 - (iii) if the population ever exceeds what the environment can sustain (called the *carrying capacity*), then the rate of change will be negative and the population will decrease.
- (a) Sketch the general shape of population vs time graphs that reflect these refined assumptions.
- (b) Explain how the rate of change equation $\frac{dP}{dt} = 0.3P\left(1 - \frac{P}{10}\right)$, which is called the **logistic model**, reflects these three features.
- (c) What does the logistic model predict if the initial population is zero? What if the initial population is 10 (think of 10 as scaled for 1000 or 10,000)?





Beginning a Numerical Approximation

2. Suppose you are using a related rate of change equation $\frac{dP}{dt} = P\left(1 - \frac{P}{10}\right)$ to investigate the change in population for a particular species of fish in a large lake and you want to numerically estimate the number of fish. The initial data (e.g., $t = 0$) says there are 2 fish in the lake (where t is in years and 2 is scaled for 2,000 or 20,000, for example).
- (a) What is the initial rate of change given this initial data and how do you interpret the meaning of this value?
- (b) Use the value of your initial rate of change to compute a numerical estimate for the number of fish in the lake 3 years later.
- (c) Try and modify your approach to get a more accurate estimate for the number of fish 3 years later.





Unit 2: A Numerical Approach

3. The goal of this problem is to graphically connect your previous work to a slope field using the Slope Field Stitcher Applet.
 - (a) Open the applet and graphically reproduce both your original and improved estimates for the number of fish after 3 years.
 - (b) Use the space below to redraw what you created with the applet. Be as detailed as possible. Also include any insights or questions you have about the relationship between vectors corresponding to different step sizes.

CONNECTION TO SECONDARY MATHEMATICS

Proportional reasoning is the ability to understand and apply relationships between quantities where the ratio between them remains constant. It involves recognizing that quantities change in a predictable way when their ratios are maintained.



- (c) Focusing only on comparing the first step of two different step sizes, describe how proportional reasoning could be used.





The Tip-to-Tail Method Numerically

4. Using the differential equation $\frac{dP}{dt} = P \left(1 - \frac{P}{20}\right)$ and initial condition $P(0) = 10$, José and Julie started the following table to numerically keep track of their tip-to-tail method for connecting tangent vectors. Explain José's and Julie's approach and complete their table.

t	P	$\frac{dP}{dt}$
0	10	5
0.5	12.5	
1.0		
1.5		

5. Using the same differential equation and initial condition as José and Julie, Derrick and Delores started their table as shown below. Explain how Derrick and Delores' approach is different from José and Julie's and then complete their table.

$$\frac{dP}{dt} = P \left(1 - \frac{P}{20}\right)$$

t	P	$\frac{dP}{dt}$
0	10	5
.25	11.25	
.5		
.75		

CONNECTION TO SECONDARY MATHEMATICS

Rate of change is a key concept in high school mathematics. How do you think about what a rate of change is? Has your thinking about rate of change changed since the first day of class?





6. (a) Consider the differential equation $\frac{dy}{dt} = y + t$ and initial condition $y(0) = 4$. Use José and Julie's approach to find $y(1.5)$. Show your work graphically and in a table of values.

(b) Is your value for $y(1.5)$ the exact value or an approximate value? Explain.

7. **Generalizing your tip-to-tail approach.** Create a recursive procedure/algorithm that would allow you to predict future y -values for any differential equation $\frac{dy}{dt}$, any given initial condition, and any time increment. Use the Next-Now notation shown below to complete the recursive procedure/algorithm.

$$y_{\text{next}} = y_{\text{now}} + \dots$$





Proportional Reasoning: Connections to Secondary Mathematics

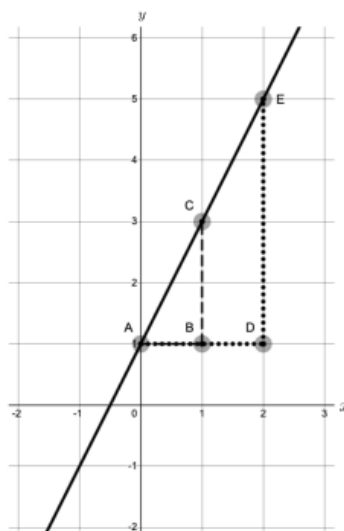
CONNECTION TO SECONDARY MATHEMATICS

You just reinvented Euler's Method and compared approximations that used different step sizes (Δt). Comparing two different approximations requires careful thinking about slope and proportional reasoning, two big ideas in secondary school mathematics.



8. In the tip to tail Euler's method, one uses a constant rate of change over a specific time increment. Graphically this is shown as a constant slope. Conceptually understanding constant slope can be challenging for secondary school students. This next problem unpacks one of these challenges.

Max is trying to determine the slope of the line graphed below. He drew in the dashed slope triangle ($\triangle ABC$) and dotted slope triangle ($\triangle ADE$) and concluded that the slope of the dotted triangle is larger than the slope of the dashed triangle because the dotted triangle has a vertical change of 4 units while the dashed triangle only has a change of 2 units.



- (a) Develop an explanation that would help Max understand that the dotted and dashed triangles represent the same constant slope.





Unit 2: A Numerical Approach

(b) Consider the following two tables of x and y values.

x_1	y_1
-3	-5
-2.5	-4
-1	-1
-0.25	0.5
0	1
2	5
2.75	6.5
4	9

x_2	y_2
-6	-11
-3	-5
0	1
3	7
6	13
9	19
12	25
15	31

- (i) Which table matches Max's line? Justify your response by calculating multiple ratios.
- (ii) What are the similarities and differences between the two tables? Explain your reasoning.
- (c) Consider the differential equation $\frac{dP}{dt} = 2$ and $P(0) = 3$ as the initial data. Compute and graph the first step of Euler's Method using $\Delta t = 1$. **On the same set of axes**, graph the first step of Euler's Method for the same differential equation but now using $\Delta t = 0.5$.
- (i) What do you notice about the ratios $\frac{2}{1}$ and $\frac{1}{0.5}$? How is that reflected in your graphs of the first step of Euler's method?
- (ii) How is your response to part (i) related to Max's graph?
- (iii) Rather than using $\Delta t = 1$ or 0.5 , we could have used $\Delta t = 0.25$ or even 0.1 . Complete the missing number in each ratio below so that they have the same slope as above.

$$\frac{?}{0.25} \quad \frac{?}{0.1}$$





Homework

Connections to Secondary Mathematics

- In class, you reinvented **Euler's Method**, which is a recursive algorithm for approximating solutions to rate of change equations. **Recursive algorithms** also appear in secondary school mathematics. One example is arithmetic sequences which also involve proportional reasoning. Consider the following two plants.

- The height of Plant A grows in a consistent pattern. Each day, Plant A's height is 2 cm taller than the previous day. Assume the initial height is 12cm.
- The height of Plant B grows in a consistent pattern. Every three days, Plant B's height is 6 cm taller than the measurement from three days prior. Assume the initial height is 12 cm.

Now consider the following two recursive algorithms:

- $y_{\text{next}} = y_{\text{now}} + 2$, $y_{\text{initial}} = 12$ (where "now" is one day and "next" is the following day)
- $y_{\text{next}} = y_{\text{now}} + 6$, $y_{\text{initial}} = 12$ (where "now" is one day and "next" is the following day)

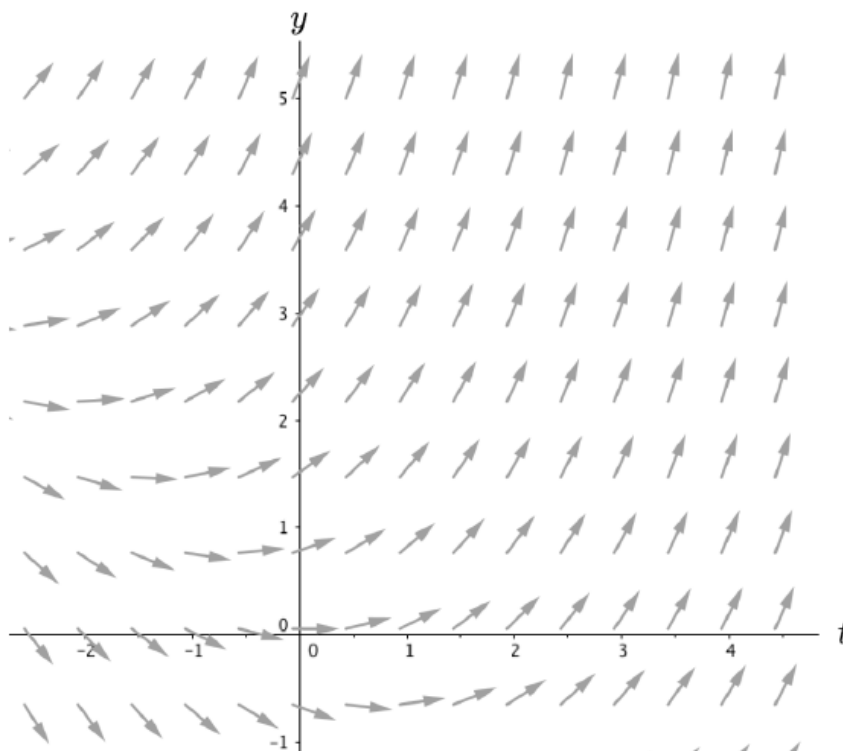
- Which recursive algorithm represents Plant A? Use proportional reasoning to explain your reasoning.
 - Which recursive algorithm represents Plant B? Use proportional reasoning to explain your reasoning.
- Two hiking trails, Trails A and B, rise at constant rates. That is, the elevation gain is proportional to horizontal distance.
 - After hiking 0.75 miles on Trail A, you have gained 540 feet of elevation. Trail A ends after 2.4 miles. What is the total elevation gain? Show your work, explain your reasoning, and provide a diagram to illustrate and clarify.
 - Trail B, gains 1200 ft in elevation over 2 miles. Which trail is steeper? Show how you know without guessing. Show your work, explain your reasoning, and provide a diagram to illustrate and clarify.
 - Recipe A calls for 4 cans of concentrate to make 6 liters of orange juice. Recipe B calls for 6 cans of concentrate to make 8 liters of orange juice. Which orange juice recipe will taste more "orangey"? Use proportional reasoning to justify your answer.
 - At the start of class you may or may not have said that one of the meanings you have for $\frac{dP}{dt}$ is that of a function. To what extent do you now think about $\frac{dP}{dt}$ as a function? Read over the three meanings of function in the Glossary and reflect on which of these meanings most closely associated with how you think about $\frac{dP}{dt}$ as a function. You are not limited to one meaning.
 - In lessons on Unit 2, when might you have engaged in Practice 8 from the Standards for Mathematical Practice? See a full list and associated definitions of all practices in the Appendix.





IODSM Content

6. A slope field for the differential equation $\frac{dy}{dt} = 0.5(y + t)$ is shown below.



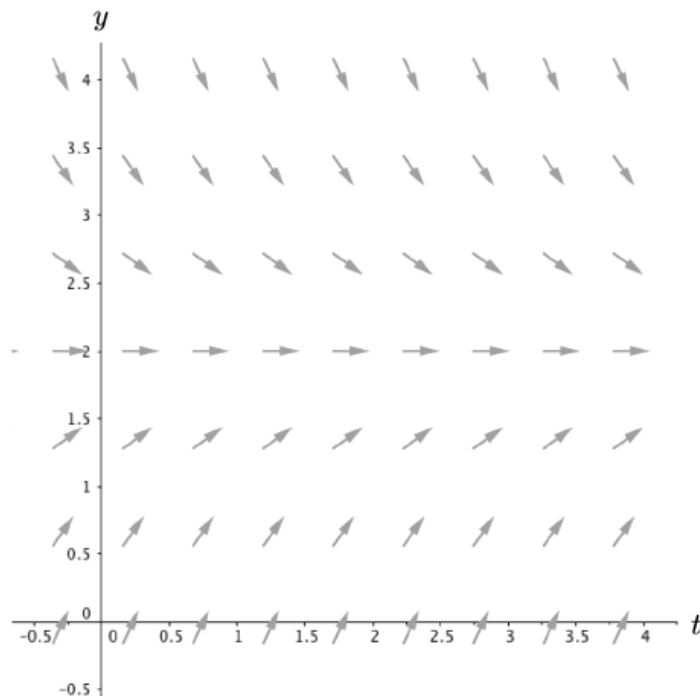
- For the initial condition of $y(0) = 1$, sketch on the above slope field what you think two iterations of the “tip-to-tail” method with a step size of 1 unit should look like. Do this **without** doing any computations.
- Again, without doing any computations, sketch on the same slope field what you think three iterations with a step size of 0.5 units should look like for the same initial condition (perhaps using a different color).
- Use the tip-to-tail (*i.e.*, Euler’s) method to numerically compute approximations for parts 6a and 6b and then compare your graphical predictions to the numerical results.





Unit 2: A Numerical Approach

7. Consider a differential equation with the given slope field and the initial value $y(0) = 1$.



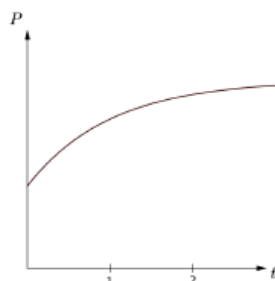
- Explain why, if you wanted to approximate $y(2)$ using two steps of Euler's method, you would need $\Delta t = 1$.
- Use a straight edge to graph two steps of Euler's method to approximate $y(2)$.
- This time, instead of using two steps of Euler's method, sketch on the same slope field what it would look like if you used four steps of Euler's method to approximate $y(2)$.
- Besides the obvious difference that the step size is different, state two other things that are different between your answers to parts 7b and 7c.
- Besides the obvious fact that they both use Euler's method, what is similar about the first step to your answers to parts 7b and 7c? *Hint: Refer back to Classwork Problem 8 about Max.*



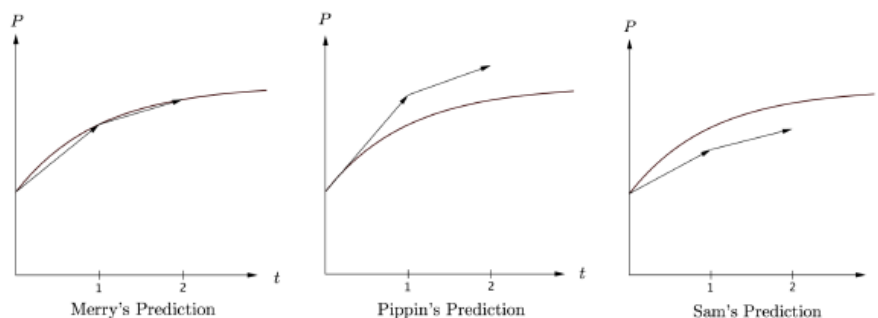


Unit 2: A Numerical Approach

8. Suppose we have a rate of change equation and initial condition for the population of raccoons in Lake County. Below is a graph of an exact solution.



Merry, Pippin, and Sam attempted to use the “tip-to-tail” Euler method to predict what the population of raccoons would be at time $t = 2$, with time increments one unit. However, they arrived at different graphs for their predictions. Their predictions are given below, and are shown with the exact solution.



For each prediction, give reasons as to whether or not each person illustrated the correct relationship between Euler's method and the exact solution.

9. Suppose the function $y(t) = 6t + 1$ is a solution to a particular differential equation. For the initial condition $y(0) = 1$, is a graph of the tip-to-tail Euler method exactly the same as the graph of the exact solution? Does your response depend on step size? Explain.
10. Compute by hand four steps of the tip-to-tail Euler method for the differential equation $\frac{dy}{dt} = y - t$ with initial condition $y(0) = 2$ and step size 0.5.





Unit 2: A Numerical Approach

11. **Euler's Method Using a Spreadsheet.** Learning to use a spreadsheet for various applications in engineering and mathematics is a valuable skill. Your task in this problem is to use a spreadsheet to generate as many steps of the Euler method that you want. *Below is an example to get you started using Excel. If you are already familiar with Excel, skip the example below and go directly to part a.*

EXAMPLE: Here are step by step instructions for how to use Excel to generate 15 steps of the algorithm $Y_{\text{next}} = 2 \cdot Y_{\text{now}} + 1$ with initial condition $Y = 3$.

- Open an Excel workbook
 - Select cell A1 by clicking on the cell in this location and type in Y_{now} as a column heading
 - Select cell B1 and create a column heading called Y_{next}
 - Select cell A2 and type in the number 3 (this is the given initial Y -value)
 - Select cell B2 and type $=2*A2+1$ (after pressing Enter the number 7 will appear in this cell)
 - Select cell A3 and type $=B2$
 - Select and copy cell B2 (An animated dashed-line will appear around the cell)
 - Select cells B3 through B15 and paste
 - Select and copy cell A3
 - Select cells A4 through A15 and paste
 - Do a few hand computations to verify the results
- (a) Using a step size Δt of your choice, figure out how to use a spreadsheet to generate at least 20 steps for Euler's method, $y_{\text{next}} = y_{\text{now}} + \left(\frac{dy}{dt}\right)_{\text{now}} \cdot \Delta t$, for the differential equation $\frac{dy}{dt} = 0.3y \left(1 - \frac{y}{12.5}\right)$ with initial condition $y(0) = 3$. In order to make it easier to graph the results, make your first column t_{now} and your second column y_{now} . Turn in a print out your results and verify the first three steps by hand.
- (b) Use the chart function (e.g., Chart Wizard scatter plot option) to create a graph of your (t, y) data from part 11a. An easy way to do this is to first highlight all the data in the t_{now} and y_{now} columns, select Chart Wizard, and follow the prompts. Turn in a print out of your results.
12. A group of scientists created the differential equation $\frac{dP}{dt} = 0.8P \left(1 - \frac{P}{5}\right)$ to predict future fish populations in Lake Minnetonka, where P represents thousands of fish and t is in years.
- (a) If you were to plot a slope field for this rate of change equation, what window for the P and t values would you use to make sure the most important features are clearly shown? Explain.
- (b) What does this rate of change equation predict about the long-term outcome of the fish population if the initial population is 2 (i.e., $P = 2$ at $t = 0$)? How about if $P = 6$ at $t = 0$?
- (c) Why are the predictions you made in part 12b reasonable (or not) for a fish population? Explain.
- (d) Carry out by hand three steps of Euler's method with a step size of 0.5 for the initial condition $P(0) = 5$.
13. (a) Go to the glossary and identify all terms that are relevant to this unit and list those terms here.
- (b) Are there other vocabulary terms that you think are relevant for this unit that were not included? If yes, list them.



Appendix 2: Mathematical Practices

Mathematical Practices (Common Core State Standards Initiative, 2010)

MPs	Brief Description
MP1: Make sense of problems and persevere in solving them	Students grasp the mathematics, make a course of action, and follow through with solving the problem, sometimes finding multiple ways to get to a solution.
MP2: Reason abstractly and quantitatively	Students reason about the context of the problem posed and the relationships of the quantities within the problem.
MP3: Construct viable arguments and critiquing the reasoning of others	Students make conjectures using definitions and the context of the problem that can be accurately backed up by evidence and then be able to effectively communicate and defend these conjectures to their peers. Students critique the reasoning of others and effectively communicate the flaws in the logic of their peers.
MP4: Model with mathematics	Students apply the mathematics to real-life situations and create or use models to represent relationships within the context of the situation.
MP5: Use appropriate tools strategically	Students figure out and accurately choose the right tools to help understand or solve mathematics depending on the situation or problem presented to them.
MP6: Attend to precision	Students communicate precisely in writing, discussing, and labelling mathematics, such as using clear definitions or specifying units consistently.
MP7: Look for and make use of structure	Students recognise patterns and relationships and use this pattern to solve problems.
MP8: Look for and express regularity in repeated reasoning	Students recognise repeated processes and generalise them in the form of shortcuts, rules, or formulas.

