Connecting Algebraic Structures and Solving Equations: Fostering Mathematical Knowledge for Teaching

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Received: September 2025 ǁ Accepted: April 2026

Whilst in theory, the connections between advanced mathematics and secondary-level mathematics are widely acknowledged as significant by the entire community of mathematics teacher educators, mathematics teachers and student teachers, there remains considerable scope for research concerning the specific nature, types, and mechanisms of these connections. Further investigation is needed into how both practising and prospective teachers may be supported in articulating and applying these links within their classroom practice. This article endeavours to contribute to this area by presenting findings from a study in which mathematics teacher educators participated in workshops, designed learning sequences for prospective teachers, and implemented them. A detailed analysis of one mathematics educator's process focused on algebraic structures and their relationship to equation solving was carried out. The findings suggest that the experience was beneficial in enabling the integration of diverse connections into practice, thereby fostering his Mathematical Knowledge for Teaching (MKT) as well as that of his students, the prospective teachers. The article elaborates on the connections rendered visible through the Mathematics Teacher Educator's (MTEs) practice and examines the influence on the development of the MKT among the prospective teachers.

Keywords • mathematics teacher education research • connections • advanced and secondary mathematics • algebraic structures • mathematical knowledge for teaching

Bridging the Disciplinary Divide in Mathematics Teacher Education

The inclusion of advanced mathematics courses within initial teacher education programmes for prospective secondary mathematics teachers (PMTs) is a near-universal feature of curricula worldwide. This practice is widely regarded as both desirable and necessary by mathematics teacher educators (MTEs), prospective teachers, and practising secondary teachers (Even, 2011; de los Ángeles et al., 2022; Zazkis & Leikin, 2010). This consensus is underpinned by the foundational construct of pedagogical content knowledge (Shulman, 1986), which posits that a profound disciplinary understanding is a cornerstone of effective teaching.

A significant and persistent problem has been documented extensively within the literature concerning the professional practices of MTEs: a palpable disconnect between the advanced mathematics taught at the tertiary level and the school mathematics these prospective teachers are subsequently required to teach in secondary classrooms (Ball et al., 2008; Wu, 2011). This disjuncture is not merely a recent observation but an international phenomenon with deep historical roots. As Klein noted more than a century ago, the graduate teacher, confronted with the reality of the schoolroom, often reverts to traditional methods, rendering their advanced university studies "a more or less pleasant memory which had no influence on his teaching" (Klein, 1908, cited in Winsløw & Grønbæk, 2014, p. 61).

Contemporary research confirms that this challenge spans diverse sociocultural and institutional contexts. Studies indicate that PMTs and practising teachers frequently fail to perceive, much less leverage, connections among advanced mathematical knowledge and the school curriculum. Empirical evidence from Turkey (Bukova-Güzel et al., 2010), the United States (Cofer, 2015; Cook, 2012; Cuoco, 2001), and Uruguay (Ochoviet, 2015) consistently demonstrates that even after completing advanced courses, PMTs struggle to apply this knowledge to explain fundamental school-level concepts. This is further compounded by findings that the number of advanced mathematics courses taken shows no strict correlation with the performance of a teacher's future students (Hine, 2018; Monk, 1994). Critically, this difficulty in articulating concrete connections is not limited to PMTs; it extends to the MTEs, including mathematicians, who often value such links in principle yet find them challenging to exemplify in practice (Yan et al., 2022). Moreover, as Suominen (2018) concluded, "It is not sufficient to simply make a list of mathematical connections but rather one must consider the ways in which these connections are discussed" (p. 196).

This body of work suggests that the core issue lies less in the presence of advanced content and more in the nature of its instruction. Ticknor's (2012) research on abstract algebra illustrated that, when the objectives of a course are confined to imparting advanced, abstract knowledge devoid of explicit contextualisation to school mathematics, the resulting learning remains as common content knowledge rather than evolving into the specialised content knowledge that teachers need (Ball et al., 2008). Ticknor (2012) explicitly stated, "There may be a need for instructors in mathematics to have intentional and explicated goals to connect the number systems and structures discussed in abstract algebra with the algebra taught at the secondary level" (p. 322).

It is within this well-established international context that the present study is situated. This research seeks to address the following questions:

How do mathematics teacher educators explicitly connect tertiary mathematics with secondary-level content through specifically designed interventions?

How does such mediation impact the connections prospective mathematics teachers are able to establish, and the development of their mathematical knowledge for teaching?

To this end, the study reports on a professional development initiative wherein MTEs collaborated with researchers to design and implement teaching sequences that deliberately bridge this gap, drawing on theoretical frameworks that privilege the development of knowledge for teaching (e.g., Wasserman, 2018a; Zazkis & Marmur, 2018). Through a detailed case study of one participating MTE, this article aims to contribute tangible tools and insights to the mathematics teacher education community, moving beyond identifying the problem, and towards evaluating a potential pathway for its resolution.

Theoretical Framework

The theoretical framework is based on two well-known constructs, mathematical knowledge for teaching and the points of connection between advanced and secondary mathematics. After brief descriptions, a relationship between them is proposed.

Mathematical Knowledge for Teaching

Shulman (1986) proposed that the knowledge required for teaching constitutes a network of various interconnected types of knowledge, which he termed: content knowledge; pedagogical content knowledge; general pedagogical knowledge; curriculum knowledge; knowledge of learners, and their characteristics; knowledge of educational contexts; and knowledge of educational ends, purposes, and values. Building on this foundation, Ball et al. (2008) drew on a detailed analysis of the knowledge mathematics teachers used in their practice to develop a framework for Mathematical Knowledge for Teaching (MKT), distinguishing between subject matter knowledge and pedagogical content knowledge. Within these two primary domains, they further identified three sub-domains each, as illustrated in the Table 1. Whilst these domains and sub-domains are closely interrelated and it can be difficult to distinguish which type of knowledge is being utilised in a teaching activity, descriptions of each category are relevant for studying the kind of knowledge important for teacher education. Hine

(2018) took the definitions of each of these subject matter knowledge sub-domains from Ball et al. (2008) and illustrated through examples pertaining to the multiplication of two numbers. His work highlighted the relevance and utility of the MKT framework.

Table 1
Domains of Mathematical Knowledge for Teaching (Ball et al., 2008, p. 403)

Subject Matter Knowledge	Pedagogical Content Knowledge
Common Content Knowledge (CCK)	Knowledge of Content and Students (KCS)
Specialised Content Knowledge (SCK)	Knowledge of Content and Teaching (KCT)
Horizon Content Knowledge (HCK)	Knowledge of Content and Curriculum (KCC)

The sub-domain Common Content Knowledge (CCK) includes the mathematical knowledge and skill used in settings other than teaching. It is the knowledge a teacher therefore holds in common with other professionals. Hine (2018) exemplified this knowledge as knowing the standard algorithm for multiplying two numbers. Specialised Content Knowledge (SCK) is the mathematical knowledge specifically required for the decisions inherent to teaching: what to teach, how to teach it, which interactions to facilitate, and which strategies to promote amongst students. For instance, knowing that the multiplication algorithm connects to place value and the distributive property is important for designing teaching situations, classroom management, guiding student thinking, and responding to questions that may arise in the classroom. Finally, Horizon Content Knowledge (HCK) refers to an awareness of how mathematical topics are related across the school curriculum and the foresight to establish connections with subsequent mathematical ideas. Continuing with the example provided in Hine (2018), this would involve an awareness that the algorithm for multiplying two numbers is related to the multiplication of two polynomials. Ball and Bass (2009) expanded this conception of HCK to also include knowledge of advanced mathematical ideas and structures, key mathematical practices, and core mathematical values and sensibilities.

Within the Pedagogical Content Knowledge (PCK) domain, Knowledge of Content and Students (KCS) refers, according to Ball et al. (2008), to the intersection of the understanding of mathematical content and knowledge of student thinking: knowing what difficulties can be anticipated in learning a specific topic, anticipating possible student approaches to a task, or judging whether an example might be motivating and relevant to foster their understanding, amongst other teaching requirements. Knowledge of Content and Teaching (KCT) integrates the understanding of mathematical content with an understanding of pedagogical aspects that influence student learning (Ball et al., 2008). KCT includes decisions about which tasks to propose, which resources to employ, what types of interactions to promote in the classroom, when a question is required and when silence is preferable, which student comments to address for whole-group discussion and which not to. Finally, Knowledge of Content and Curriculum (KCC) was defined by Shulman (1986) as the familiarity with all curriculum programmes for a subject and their interrelations, along with knowledge of the resources and instructional strategies suggested by these programmes.

Points of Connection Between Advanced and Secondary Mathematics

Wasserman (2018a) proposed a non-exhaustive list of four types of connections between advanced and secondary mathematics, spanning a spectrum from mathematical content to pedagogical practice. *Content connections* refer to links between topics in advanced mathematics courses and those in secondary mathematics courses; *disciplinary practice connections* are instances in which PMTs engage in practices that involve doing mathematics, such as proving, refuting, defining, and classifying, and these practices are made explicit by the MTE; *classroom teaching connections* are instances where the knowledge mobilised in advanced mathematics courses serves as a resource for motivating specific types of pedagogical actions in the classroom; and *modelled instruction connections* refer to the intentional modelling of specific types of teaching. "Having teachers learn mathematics in particular ways can shape their own approaches to teaching" (Wasserman, 2018a, p. 9). This categorisation provides a broad framework for addressing the research questions by highlighting that the connections extend beyond mathematical content to include disciplinary activities, classroom management, and modes of teaching.

Establishing Connections Between Advanced and Secondary Mathematics as a Framework for Developing Mathematical Knowledge for Teaching

This article proposes that the establishment of connections between university-level mathematics and secondary school mathematics by MTEs contributes in multiple ways to the development of MKT for both MTEs and PMTs (Dellori & Wessel, 2023; Wasserman, 2018a, 2018b; Wasserman et al., 2017; Wasserman et al., 2019). This section presents the foundation for this hypothesis, which is subsequently tested through the analysis of the intervention described.

Content connections contribute to the development of SCK, as they provide PMTs with a deeper perspective on topics that may seem basic, facilitating informed decision-making about what to teach and why. Furthermore, this type of connection contributes to HCK, as it broadens the PMTs' view of how the content they teach at one level relates to content students will encounter later, or within the same course. In this sense, these connections reveal the underlying structure of secondary school content, providing teachers with a broader horizon for their professional practice. Content connections also foster the development of KCC, as they make visible the relationships between the content of different curricula at different levels.

Disciplinary practice connections contribute to SCK; for example, the mathematical practice of analysing a proof or line of reasoning is necessary for the teacher's practice of analysing student procedures and for anticipating student responses. Similarly, the practice of defining a mathematical object provides tools for making decisions about which definition might be appropriate in each teaching context. In turn, this type of disciplinary practice connection contributes to HCK, as these practices are necessary for developing the horizon mathematical knowledge that extends beyond what is explicitly taught in secondary courses.

Classroom teaching connections also contribute to SCK, for instance, for evaluating student conjectures, anticipating unusual solution methods (Castro & Li, 2014), finding straightforward ways to make mathematics manageable for novices, designing tasks, seeing underlying connections in school mathematics topics, and extracting ideas from students (Conference Board of the Mathematical Sciences (CBMS), 2012). Likewise, the connections also apply to KCT by providing input for designing tasks that highlight the internal structure of mathematics, formulating questions for specific teaching purposes, or selecting the most suitable representation of an object for addressing a topic in a specific teaching situation. This type of connection also contributes to KCS, as it enables the interpretation of a student's unexpected question or error through a deeper mathematical lens.

Finally, establishing modelled-instruction connections involves the enacting of teaching practices that future teachers are expected to develop in their practice. Although this connection point relates to all sub-domains of MKT, it contributes particularly to KCT and KCS, for example, through selecting and designing tasks that make student thinking and productive conversations visible. The proposed relationships among the development of connections between advanced and secondary mathematics (Wasserman, 2018a), on one hand, and mathematical knowledge for teaching (Ball et al., 2008), on the other hand, are illustrated in the following diagram (see Figure 1). The ideas of nonlocal mathematics (Wasserman 2018b) and points of connection between advanced and secondary mathematics (Wasserman 2018a) were developed taking the MKT framework developed by Ball et al. (2008) as a starting point—a possible framework for describing, interpreting, and analysing the mathematical knowledge required for teaching. This schema illustrates is that the types of connections can serve to operationalise the sub-domains of MKT; that is, they are mechanisms that MTEs can utilise to foster the development of the various sub-domains necessary for practising and prospective mathematics teachers.

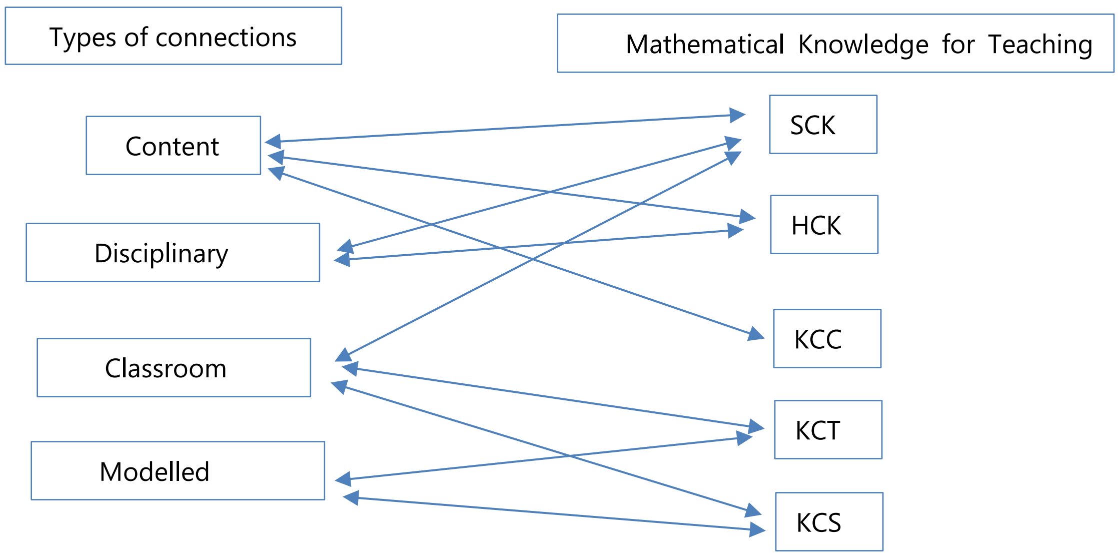


Figure 1. Relationships among types of connections between advanced and secondary mathematics (Wasserman, 2018a) and MKT (Ball et al., 2008).

Methodological Aspects

Method and Participants

The research development comprised an initial stage of two workshops with MTEs—lecturers teaching mathematics courses within mathematics teacher education programmes. A broad invitation was extended to all MTEs from various teacher training institutes across the country. Nine MTEs from different institutes participated voluntarily in the workshops.

During the workshops, the issue of the lack of connections between advanced and secondary-level mathematics was presented. Research findings related to this issue were shared, highlighting its prevalence among MTEs, PMTs, and in-service teachers, all of whom struggle to identify concrete and explicit links between their university studies and teaching practice. Examples of potential connections were provided, along with findings from studies that have implemented learning sequences for both practising and prospective teachers, taking these connections into account. The MTEs were encouraged to identify explicit connections between the advanced mathematics taught in their courses and secondary-level mathematics. To support this task, they were provided with the lower secondary education curricula (Levels 7 to 9, Administración Nacional de Educación Pública [ANEP, 2023]). Finally, the possibility of collaborative planning and implementation of a teaching sequence for a mathematics topic within their courses was introduced. This work was carried out individually or in pairs with those MTEs who expressed interest in the proposal.

Both workshops were audio-recorded and transcribed, using pseudonym for anonymise the data. The MTEs' conclusions regarding explicit connections between advanced and secondary mathematics were documented in real time by the researcher (see Molfino, 2023). These data were analysed along tow lines: the first examined, across the group of MTEs, the types of connections established and the MKT mobilised, whilst the second focused on each MTE's individual responses and contributions during the workshop.

The second stage of the fieldwork involved the planning of teaching sequences to be implemented in the MTEs' courses. Four MTEs expressed interest in the proposal. Two of them worked individually—one on a geometry course and the other on an algebra course—while two others worked in pairs to design a sequence for an algebra course. All three sequences were developed collaboratively with the researcher, and the process recorded in subsequent written versions of the teaching sequence plans.

The third stage of the research comprised of the implementation of the sequences planned by the MTEs in their respective courses, with the researcher participating as an observer during the sessions. These sessions were audio-recorded and transcribed, and the PMTs written productions were collected at the end of each session. As well, pictures were taken of the notes made on the board by the MTEs. The analysis of the PMTs' productions and class discussion focused on two areas: the development of the PMTs' MKT, and the moves made by the MTEs, which revealed the types of connections they fostered and the MKT the connections mobilised. The fourth and final phase involved a post-implementation interview by email, which explored the impact of the workshop-design-implementation module on the MTEs. The written responses were recorded and analysed using the same conceptual framework lenses.

While Molfino (2026) reported on the geometric sequence implementation by focusing on the PMTs' work, this article presents an in-depth analysis of the case of George, one of the four MTEs who completed all research phases (workshops, planning, implementation, and interview). The case of George was selected for an in-depth analysis due to the richness and depth of the data he provided. His high level of engagement and reflective disposition during the post-implementation interview yielded particularly detailed and nuanced accounts of his experience. This depth makes his case especially fertile for examining the processes through which an MTE interprets and integrates connections between advanced and secondary mathematics across all phases of the professional development cycle.

MTE George graduated from the same institution where he was working as MTE. He holds a master’s degree and has four years of experience as MTE. The course for which the sequence was designed was a first-year algebra course within the mathematics teacher education programme. The selected topic, chosen by the MTE, was algebraic structures, specifically group properties. George opted to design a sequence based on adapted activities from Murray and Baldinger (2018), which had been presented during the training workshops.

Methodological Model for Teaching Sequences Design

The design of the teaching sequences was informed by the methodological model proposed by Wasserman et al. (2017, 2019). Drawing on Ticknor (2012), this model emphasises a situated perspective on teaching, in which the mathematical resources teachers draw upon in their professional practice are shaped by the learning experiences they encounter during their training. The model is comprised of three phases (see Figure 2).

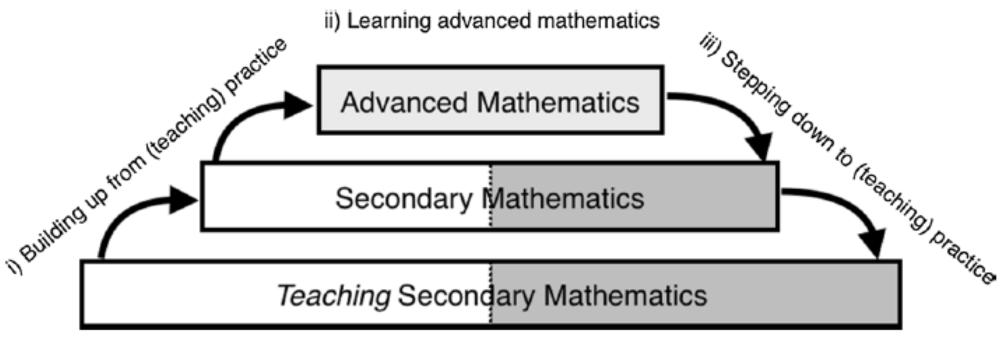


Figure 2. Methodological model by Wasserman et al. (2019, p. 386).

The teaching sequence designed in the case reported in this article addressed connections between the teaching of equations and the learning of algebraic structures. It was implemented by George teaching a group of 25 PMTs in 'Fundamentos 2', a first-year, semester-long mathematics course. The intended content connection involved linking properties of the real number field—used in solving equations at the secondary level—with algebraic structures, a topic explicitly included in the 'Fundamentos 2' syllabus. The sequence involved the resolution of non-routine problems within algebraic structures not typically encountered in school mathematics and identifies mathematical structures to support procedures and apply them to solve these problems, thus establishing disciplinary practice connections. The first task of the sequence invited PMTs to analyse secondary school students' equation solving strategies and give feedback, promoting classroom teaching connections. Finally, the nature of the proposed tasks and their implementation—open ended tasks, placing PMTs at the centre of knowledge construction, discussion in small and large groups—also aimed to foster modelled instruction connections.

The sequence was inspired by a task presented in Murray and Baldinger (2018) and adapted to the course by the MTE in collaboration with the researcher. In Phase (i), PMT were asked to analyse a solution to an equation provided by a Year 9 student, identify the properties the student may have invoked, and comment on the reasoning. Phase (ii) was addressed through an activity that involved solving equations within non-standard algebraic structures: $(\mathbb{Z}_3,+,\bullet)$ and $(\mathbb{Z}_4,+,\bullet)$. This activity provided a context for the MTE to institutionalise knowledge of algebraic structures with the degree of rigour expected at tertiary level. A scripting task (Zazkis & Marmur, 2018) was designed to achieve Phase (iii). It required PMT to synthesise their reflections from the previous phases by continuing a fictional dialogue between a teacher and secondary students. Wasserman et al. (2019) argued that starting and concluding with scenarios set in secondary classrooms is essential for developing the situated knowledge of teaching that future teachers require.

Analysis of Findings

This section presents the analysis in three parts, corresponding to the methodological stages. The first part (Stage 1) examines the interactions among MTEs during the workshop sessions. The second part (Stages 2 and 3) focuses on the outcomes of the PMTs as they implemented the instructional sequence in the classroom, alongside an analysis of the MTE's role in orchestrating discussions and fostering PMTs collaboration. The final part (Stage 4) investigates the reflective practices of the MTE, as captured in a post-implementation interview.

Workshop Insights and Design Considerations

The initial workshop for teacher educators served to raise awareness regarding the core thematic concerns. Numerous observations from the literature review were corroborated during the session: while the educators recognised the significant value of establishing explicit connections between the mathematics taught in their teacher preparation courses and secondary-level mathematics, they demonstrated limited ability to articulate these links with precision (de los Ángeles et al., 2022; Yan et al., 2022; Zazkis & Leikin, 2010).

When asked to justify the importance of such connections, the MTEs emphasised the need to reinforce conceptual understanding through in-depth study of secondary topics, to establish meaningful links between content areas, and to develop criteria for prioritising content within a syllabus. They also highlighted the value of presenting secondary mathematics content as specific instances of more general mathematical objects or properties, providing both examples and non-examples, and acknowledged an affective component: enhancing teachers' confidence in their teaching.

These reasons for incorporating advanced mathematics in teacher preparation align with the first rationale proposed by Wasserman (2018a, p. 2): "the advanced ideas discussed in these courses are foundational for the mathematics studied at the secondary level." However, the second rationale cited by the author did not emerge in their responses: "the study of advanced mathematics is productive for instilling broader mathematical norms, sensibilities, practices, etc., in teachers; for developing a sense of what 'doing mathematics' entails" (Wasserman, 2018a, p. 2). In terms of Even's (2011) results, it is clear that MTEs considered studying of advanced mathematics as a key aspect in the teacher mathematics education but could appreciate just one of the dimensions-knowledge of mathematics itself. They did not consider the dimension regarding knowledge about mathematics, that is, a general understanding of the nature of the discipline.

In particular, George agreed on the necessity of teaching advanced mathematics but initially showed difficulty in making connections explicit:

> I do see that there are elements in those courses [within the teacher preparation programme] that are connected to what is taught at the secondary level—for example, compactness. It's like an extension of what is covered in secondary school. There are connections, even if it is 'advanced mathematics'. The challenge is to make explicit the connections that exist between advanced mathematics and secondary-level mathematics. [George, Workshop 1]

He justified the importance of teaching advanced mathematics through the need to establish links with secondary content as an end in itself, rather than emphasising the intrinsic value of studying advanced mathematics.

During the workshop, several MTEs indicated that a significant barrier to implementing such connections was time constraints, noting insufficient room within their existing syllabi to incorporate these elements. However, some MTEs contended that it was not solely a matter of time, but rather one of pedagogical prioritisation: what content is emphasised and how? Underlying their discussions was also a difficulty in situating themselves within the secondary curriculum, particularly for those not currently teaching at that level.

To address this, the workshop included a structured review of the secondary mathematics curriculum (Years 7–9, ANEP [2024]). MTEs were provided with printed curriculum documents and asked to identify specific connections with their own course content. This exercise proved fruitful: within a short time, the MTEs identified three specific links:

- The various representations of rational numbers and their connection to the study of number sets and group properties in the Foundations course.
- Equations of lines and other topics in analytic geometry linked to the Geometry and Linear Algebra course.
- The study of isometries as functions (via their mapping rules) in secondary school, and the analysis of the group formed by the set of isometries under composition in the Geometry course.

None of these specific connections were contributed by George; they were proposed by other workshop participants.

In a second workshop, MTEs were presented with a series of activities drawn from the literature (Cook et al., 2023; Murray & Baldinger, 2018; Zazkis & Marmur, 2018), serving as illustrations of the types of tasks intended to foster connections between advanced and secondary mathematics. The educators engaged actively with the tasks and considered them relevant for inclusion in their own courses.

Discussions about these examples prompted initial developments in George's MKT. The course he teaches is a first-year algebra module covering the real number system, including the study of natural, integer, and rational numbers. Reflecting on this, George identified a weakness in his current approach:

> The description of number sets is a topic not covered in Foundations [the name of the algebra course]—we only address it through the axioms of a field. Teaching it at the secondary level would create a strong connection if it were approached from the perspective of construction [of number systems], rather than axiomatically (George, Workshop 2).

This reflection may indicate development in both SCK and HCK, stimulated by considering the connections between advanced and secondary mathematics. It demonstrates a critical understanding of how different mathematical approaches can either support or hinder future teachers' classroom practices.

The work conducted in the workshops led to the development and implementation of three distinct instructional sequences: one in geometry, focusing on the disciplinary practice of proof within the context of axial symmetry; and two in algebra, one concerning rational numbers and another addressing the practice of equation-solving in relation to algebraic structures. The last of these, which is presented in the next section, was developed by George in collaboration with the researcher.

Classroom Implementation and Emerging Outcomes

The instructional sequence designed collaboratively by George and the researcher was implemented over a 120-minute class session within a first-year algebra course -Fundamentos 2-, in charge of the MTE.

Phase (i): Building up from (teaching) practice

In the first activity, PMTs were asked to identify the mathematical properties used by a middle-school student in the following procedure (see Figure 3):

$3x^2 - 6x - 24 = 0$

"I added 24 in both sides and divided by 3".

$3x^2 - 6x = 24$

$x^2 - 2x = 8$

$x.(x-2) = 8$

Figure 3. Extract of Activity 1.

The 25 students worked in small groups. Three types of responses were identified. Some groups—comprising 15 of the 25 PMTs—incorrectly stated that the properties used were monotonicity under addition, monotonicity under multiplication, and distributivity of addition in relation to multiplication, as illustrated in Figure 4.

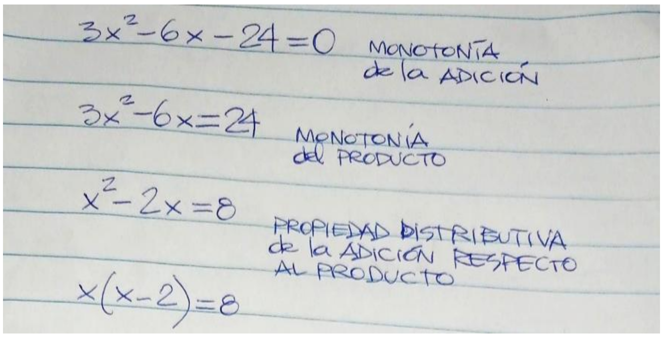


Figure 4. Example of a written PMT's response to Activity 1.

Given that the PMTs had studied the axiomatic system of real numbers in a course only two months earlier, this prevalent error suggests a disconnect between their formal knowledge and their ability to analyse a secondary pupil's solution. One possible explanation is that the PMTs applied a property valid in one context (e.g., inequalities in ($\mathbb{R}$, +, ·; ≤)) to an inappropriate context. Another possibility is a vague recollection of properties mentioned during their own secondary education.

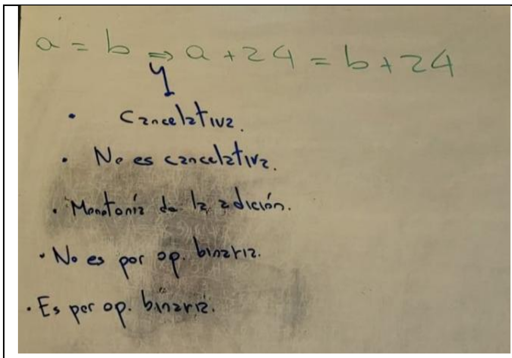
The second type of response correctly referred to group properties such as the existence of additive inverses and the distributive property, though these explanations only partially accounted for the steps in the pupil's solution. Finally, only one subgroup accurately and comprehensively identified all relevant properties.

The responses were then discussed collectively, with George mediating to reach a correct and complete answer. He began by asking: "What mathematical properties did the pupil use to solve this equation?" and listed the PMTs' responses on the board (see Figure 5):

- Monotonicity of addition
- Existence of the symmetric element
- Monotonicity of multiplication
- Existence of the inverse
- Distributivity of multiplication over addition
- Factorisation (using the distributive property)
- Existence of the identity element
- Cancellation property (instead of monotonicity)
- Binary operation in ℝ (instead of monotonicity of addition)

Figure 5. PMTs' responses to Activity 1 written on the board by the MTE.

Some answers were correct, others incorrect, and some were comments PMTs made to correct one another. By writing all responses on the board, George encouraged discussion and self-correction among the PMTs without direct intervention. He then isolated the first step of the solution and posed a related question in a different context: "What mathematical property justifies this deduction?" (George, class implementation). George listened to PMTs' answers and wrote them on the board (see Figure 6):



Translation:
$a = b \Leftrightarrow a + 24 = b + 24$

- Cancellation property
- It is not the cancellation property
- Monotonicity of addition
- It is not about binary operation.
- It is due to the definition of binary operation.

Figure 6. Synthesis of PMTs' answers to the question posed written on the board by the MTE: "What mathematical property justifies this deduction?

Rather than explaining the answer, George adopted a facilitative approach: encouraging PMTs to reason through the problem. As well as in the other intervention, George wrote all the answers provided by PMTs, correct or not, in order to foster the reflection over them. This enabled the PMTs to arrive at an accurate and complete answer. Some initially suggested the cancellation property, but closer inspection led others to argue that it was the converse of cancellation. After collective discussion, the PMT and George concluded that the step relied on the definition of a binary operation—specifically, the uniqueness of the image of an ordered pair under the operation.

Phase (ii): Learning advanced mathematics

For the second phase of the methodological model, PMTs were asked to discuss the following Activity 2 (see Figure 7), again within the same small groups.

In the set H = {0,1,2} we define the operations +:HxH→H and • : HxH → H such as:

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

•	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

In the set T = {0,1,2,3} we define the operations +:TxT→T and • : TxT → T such as:

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

•	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

Solve the following equation in (H,+,•) and (T,+,•). Indicate the mathematical properties used in each step.

$$2 \bullet (x+1) + 1 = 2$$

Figure 7. Activity 2 of the sequence.

The PMTs employed four strategies: applying the distributive property (which led to an incorrect solution), substituting elements from the given sets to test the equation, using the binary operation tables in a reverse-solving approach, and applying properties of algebraic structures (e.g., uniqueness of the image, existence of inverses and identity) deduced from the tables. Some PMTs combined strategies. Figure 8 illustrates a resolution by substitution of each element in H. There is a mistake in the third conclusion from PMTs ("1 is not a solution", instead of "2 is not a solution"), but it could be dismissed for the purpose of the study, because they finally concluded "S = {1}", indicating they considered 1 as a solution, and 2 as a number that does not belong to the solution set:

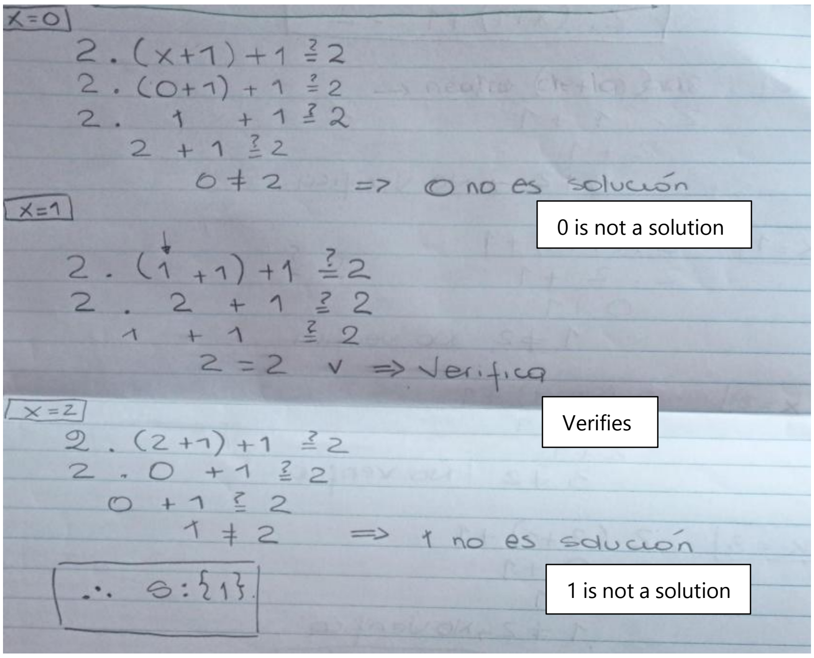


Figure 8. PMT's resolution of Activity 2 by substitution.

Figure 9 shows another resolution, applying properties of algebraic structures to solve the equation:

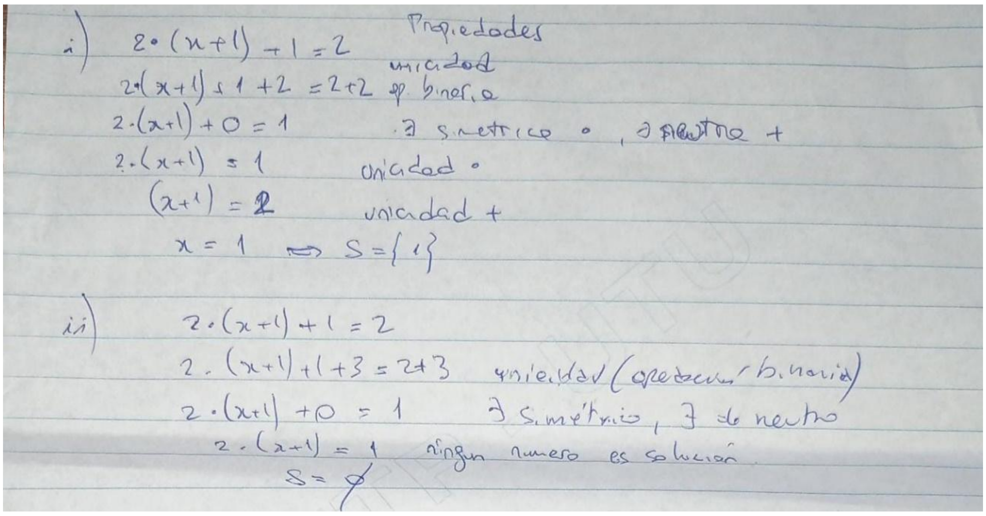


Figure 9. Written PMT's resolution of Activity 2 by applying properties.

This diversity of strategies and the emphasis on properties likely reflects the reflective practices fostered by the MTE in the previous activity. George led a collective discussion in which subgroups presented their solutions. He asked clarifying questions to encourage explicit reasoning, helping PMTs identify precisely which properties were used at each step and whether they were valid in the given context. One subgroup mentioned using the cancellation property, prompting the MTE to ask whether this property is valid in any algebraic structure. This led to a review of prior course content: cancellation is valid in a group, provided the structure satisfies associativity, existence of an identity, and inverses. The PMTs verified these properties orally for (H, +) and (H, ·) and agreed that cancellation was applicable.

During the comparison of strategies, it became evident that while some subgroups correctly used properties of (H, +) to solve the first part of the equation, they did not extend this reasoning to (H, ·). Instead, they used substitution or reverse reasoning from the operation tables. One PMT remarked, “In (H, ·) the properties don’t hold”, to which another added, “0 has no inverse in (H, ·)”. This sparked a discussion that concluded with the recognition that 0 is the additive identity and therefore not required to have a multiplicative inverse. Another PMT noted that to operate on both sides of the equation in (H, ·), associativity must be verified. This allowed the group to continue applying properties and eventually solve the equation. MTE George guided the discussion and recorded the reasoning on the board (see Figure 10):

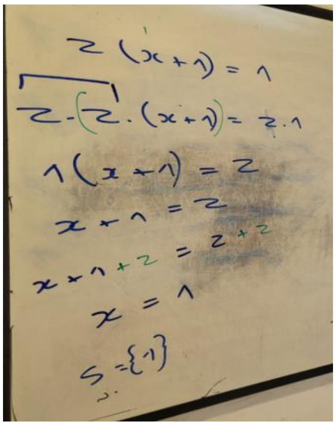


Figure 10. MTE’s board notes during discussion of Activity 2.

The researcher then asked the PMTs how they would solve the equation $2\cdot(x + 1) + 1 = 2$ in $(\mathbb{R}, +, \cdot)$. One PMT responded: "uniqueness of the binary operation, add the opposite of 1 to both sides, existence of the opposite …" leading to $2(x + 1) = 1$. When asked how to proceed, another PMT suggested, "Multiply both sides by the inverse of 2, which is ½." The researcher asked whether this is how they would teach it at the secondary level, prompting laughter and the admission that many would say *divide both sides by 2*. The researcher went on to emphasise that *dividing by 2* implicitly relies on the existence of the multiplicative inverse and the uniqueness of the binary operation. This observation aimed to help PMTs recognise that simplifying an equation involves multiplying by the inverse, and that inverses do not exist for all elements or operations. This intentional instructional move connected algebraic structures to equation-solving and modelled for George how to make advanced and secondary mathematical connections explicit, thereby supporting PMTs' development of SCK and HCK.

Phase (iii): Stepping down to (teaching) practice

Activity 3 was the last activity of the sequence (see Figure 11), which was designed according to Phase (iii) of the model (Wasserman et al., 2017). PMTs were asked to solve it in the same small groups than the previous activities.

Activity 3

Given below is the beginning of a conversation between a teacher, Pablo, and two students, Tamara and Francisco. Your task is to extend this imaginary interaction in a form of dialogue between the teacher and students. Include explanations and/or examples that you would use.

Prof. Pablo: Class, today we're going to solve $3(x - 2)2 = 6(x - 2)(x + 5)$. Who has an idea about how to start?

Tamara: I was thinking about dividing both sides by 3.

Prof. Pablo: Ok, if we do that, what will we get?

Tamara: We'd get $(x - 2)2 = 2(x - 2)(x + 5)$.

Prof. Pablo: Ok. What next?

Tamara: Now we could divide both sides by $(x - 2)$.

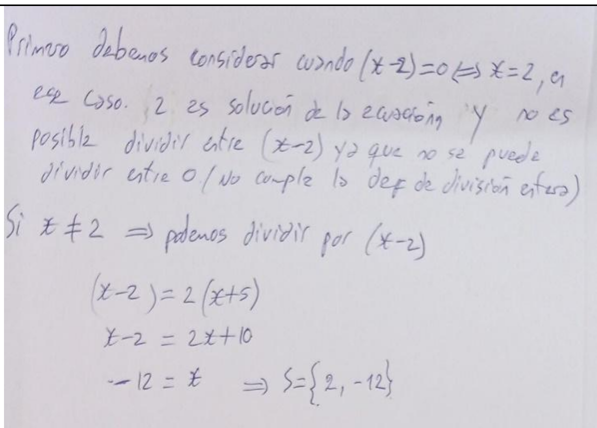
Francisco: You can't divide by $(x - 2)$.

Tamara: Why not? If you can divide both sides by 3, why can't you divide by $(x - 2)$?

Figure 11. Activity 3 of the sequence.

PMTs were presented with a fictional classroom dialogue about solving the equation $(x - 2)^2 = 2(x - 2)(x + 5)$, in which the issue of whether one could "divide both sides by $(x - 2)$" arose. None of the PMT groups applied insights from the previous activities—specifically, that simplification requires using the inverse property within the multiplicative structure of $\mathbb{R}$. While all groups addressed the invalidity of dividing by zero, none linked this to the non-existence of the inverse for certain values of x. Most argued that "you cannot divide by 0" without connecting this to the concepts discussed in Activities 1 and 2.

The resolution from the only subgroup that provided an explanation is shown in Figure 12. The solution was algebraically correct but was based on the argument that division by zero "does not satisfy the definition of integer division," rather than on the properties of algebraic structures as explored previously. These PMTs had incorrectly mentioned, in Activity 1, the monotony property as the property used by the student, but in Activity 2 were able to detail explicitly the properties of algebraic structures to solve the equation in the prompt. They do not however, employ the structures in a contextualised practice situation within a secondary school classroom. This may suggest a difficulty in transferring insights from numerical to algebraic contexts, a consideration for future sequence design and teaching strategies.

<table>
<tr>
<td>
Primero debemos considerar cuando $(x-2)=0 \iff x=2$, en ese caso. 2 es solución de la ecuación y no es posible dividir entre $(x-2)$ ya que no se puede dividir entre 0 (no cumple la def de división entera)
Si $x \neq 2 \Rightarrow$ podemos dividir por $(x-2)$
$(x-2)=2(x+5)$
$x-2=2x+10$
$-12=x \Rightarrow S=\{2,-12\}$</td>
<td>Translation:

First, we have to consider when $(x-2)=0 \Leftrightarrow x=2$, in this case, 2 is solution of the equation and it is not possible to divide by (x-2) because it is not possible to divide by 0 (it does not satisfy division in Z definition).

If $x \neq 2$ then we can divide by (x - 2)
$(x - 2) = 2(x + 5)$
$x - 2 = 2x + 10$
$-12 = x \Rightarrow S = \{2, -12\}$</td>
</tr>
</table>

Figure 12. Written resolution of a subgroup to Activity 3.

Regarding the Interview

With regard to the implications for George, a subsequent interview suggested that the experience was influential in fostering reflection and a change in teaching practices, leading him to focus more closely on the implications that the content addressing in the teacher education course had for secondary-level mathematics.

Two weeks after the implementation, a conversation was held with George, during which he remarked, "Your research resonated deeply with me, because now I pause and am very careful with details of that 'new' advanced mathematics." This suggests that George began to conceptualise a change in the very object of his teaching—there was a new perspective on what advanced mathematics represents, situated within the context of teacher education.

Furthermore, George was able to articulate a new connection not previously acknowledged:

> For example, today I worked with the arccosine function and made sure they understood that it is not the inverse of the cosine function—what implications arise from holding such a conceptual error? I provided examples and spent the entire lesson on this. What I mean is that, although I was also careful before, I now see things from a different standpoint. Previously, I might have done the same, but with less emphasis on the implications of conceptual errors in secondary school mathematics. [George, post-implementation interview]

The connection between the content being addressed—the arccosine function—and its implications for the development of secondary-level mathematics constitutes a classroom teaching connection explicitly made by George in a subsequent lesson, closely following the implementation of the designed sequence. This connection may contribute to the development of PMTs' SCK.

Lastly, George's discourse also reflected a shift in his instructional approach, which could be regarded as an instance of modelled instruction connection. The comment, "I devoted the whole lesson to it" suggested that George considered it valuable to dedicate more time to such discussions and to fostering student understanding.

Discussion and Conclusions

This study set out to investigate how the deliberate establishment of connections between university-level mathematics and secondary school mathematics by MTEs contributes to the development of MKT for both MTEs and PMTs. The findings from the design experiment, comprising workshops, collaborative planning, classroom implementation, and reflective interviews, provide compelling evidence for the fruitfulness of this approach.

Regarding Connections Between Advanced and Secondary Mathematics

The initial workshops confirmed a well-documented phenomenon in the literature (de los Ángeles, 2022; Zazkis & Leikin, 2010; Ticknor, 2012): while MTEs universally acknowledge the importance of addressing connections among advanced and secondary mathematics, they often struggle to articulate them with precision. Their initial justifications were primarily grounded in the first rationale presented by Wasserman (2018a)—that advanced mathematics is foundational to secondary content. However, the second, equally crucial rationale—that advanced mathematics instils broader mathematical norms and sensibilities for doing mathematics—was notably absent from their discourse. This highlights a critical area for development in teacher education: moving beyond a transactional view of advanced mathematics as mere content to a transformational view of it as a means to develop a deeper mathematical mindset.

Furthermore, a certain predisposition among MTEs to consider exclusively content-based connections is notable. Indeed, this is what they prioritise when justifying the importance of establishing connections between advanced and secondary-level mathematics: reinforcing conceptual understanding, studying topics in depth, generating theoretical input for decision-making regarding which content to teach, and viewing secondary-level content as specific instances of more general mathematical objects. When three connections were successfully established during the workshop, they also referred exclusively to content-based connections.

This does not, however, mean that the PMTs consider other types of connections unimportant; rather, it may be due to not deeming them relevant to mention. It is also possible that the very question—"what connections can be established between advanced and secondary-level mathematics?"—may lead to thinking solely about content, rather than the mathematical practices developed or the teaching practices employed. This could explain why the second reason reported by Wasserman (2018a)—regarding developing a sense of "doing mathematics"—and the second dimension reported by Even (2011)—understanding about the nature of the discipline, does not emerge as arguments for the importance of establishing such connections. When the four types of connections (Wasserman, 2018a) were addressed in the second workshop, the MTEs agreed in considering the other three types equally important. That is, the workshop sessions functioned as a formative instance not only to approach the issue, but also to broaden their perspective on what is meant by "connections."

In particular, the interventions by George demonstrate a development in perspective on the subject: during the first workshop, George was unable to explicitly articulate any connections, despite being convinced of their importance. By the second workshop, however, George was reflecting critically on his practice, specifically in relation to the axiomatic approach to the topic of "number sets" in his teaching and how this may hinder the development of PMTs' SCK.

Regarding the implementation of the sequence, designed collaboratively by George and the researcher and based on specific literature addressing this issue (Murray & Baldinger, 2018), George demonstrated a specific intentionality to make explicit all four types of connections. In terms of content connections, it was observed that in the first activity—which involved a "bottom-up" approach from practice—George not only used the concrete example from the task but also proposed another typical secondary-level task. This required PMTs to identify content covered in their course, specifically the properties of operations in $\mathbb{R}$, as necessary to justify the equation-solving procedure. In the second activity, George again brought this connection to the fore, but this time within the broader context of algebraic structures unfamiliar to PMTs, analysing the differences and similarities between the properties that hold in these structures and those of the well-known real number system with addition and multiplication.

George also fostered disciplinary practice connections by proposing a task that required solving an equation in a novel context, thereby engaging in problem-solving strategies characteristic of mathematical activity. Notwithstanding time constraints, which precluded explicit discussion with the PMTs, the experience nonetheless represented a valuable opportunity for them to experience it firsthand as learners, thereby furnishing a model they can replicate in the future.

As for classroom teaching connections, these were incorporated firstly into the very design of the sequence, through the selection of tasks that encouraged PMTs to identify the properties required in equation-solving. Furthermore, during the lesson in which the activity sequence was implemented, George employed interventions that promoted open discussion of ideas among PMTs, using questions that demanded the elaboration of mathematical arguments.

Finally, George also established modelled instruction connections. These were evident in the dynamics of group work, allowing sufficient time for PMTs to discuss amongst themselves, followed by a whole-class discussion that generated productive conversation based on their subgroup reflections. When PMTs presented incorrect answers, George, instead of correcting them directly, opted to propose a similar but simplified situation. This allowed PMTs to reflect on it and subsequently reanalyse their initial response to the original problem. These specific moves were experienced by PMTs in this instance as students and may be adopted for their future practice as teachers.

The classroom implementation revealed an initial disconnect among PMTs. Despite recent formal study of the axiomatic structure of the real numbers, many defaulted to misapplied concepts like monotonicity or vague procedural memories from their own schooling when analysing a pupil's solution. This suggests that without explicit pedagogical intervention, knowledge of advanced mathematics remains inert and does not automatically inform the analysis of secondary mathematics teaching practices. The successful shift in strategies observed in Activity 2—from mechanical application of algorithms to the application of algebraic properties—demonstrates that PMTs can leverage their advanced knowledge when the instructional sequence is carefully designed to provoke such reasoning.

Regarding the Mathematics Knowledge for Teaching Development

The established connections during the activities fostered incipient development of the PMTs' MKT, which was evidenced by the evolution in their responses between the activities of Phase (i) "building from teaching practice" and phase (ii) "teaching advanced mathematics" (Wasserman et al., 2019, p. 386). PMTs deepened their knowledge of the properties of algebraic structures (CCK), of how these properties underpin the solving of equations in both familiar and unfamiliar algebraic structures (SCK), and of how certain properties that hold in the set of real numbers with the usual operations do not hold in other structures (HCK). This development was supported by class discussions and is further evidenced by the variety of different solution paths the PMTs proposed for the second activity, along with their collective examination of the consequences of each approach—detailed in the section *Analysis of Findings*—which ultimately led them to conclude and generalise about properties of algebraic structures.

In the first activity, PMTs had the opportunity to reflect on which properties underpin the solving of an equation within a familiar context. However, the design pivot in the second activity, which proposes solving equations within non-standard contexts and with specific constraints, presented the students with a novel challenge. This shift triggered them to draw new conclusions about the practice of solving equations and its grounding in the properties of the relevant algebraic structures. By undertaking this pivot, PMTs were able to experience, as learners themselves, the kind of cognitive engagement they are expected to foster in their future students. This was intended to contribute to two further subdomains of MKT: Knowledge of Content and Teaching (KCT), through the design of challenging tasks, and Knowledge of Content and Students (KCS), by generating an awareness of difficulties that secondary school students might encounter when solving equations.

It was, however, observed that many PMTs who successfully identified properties in a secondary-level pupil's solution and applied them to solve equations in unfamiliar algebraic structures, did not incorporate these same properties into their explanations in the fictional classroom scenario. This suggests a need for further work on making such connections explicit so that PMTs integrate their university-level mathematical knowledge into their future professional practice. The transfer of knowledge is neither automatic nor immediate. This points towards the necessity of designing instructional sequences that not only present these connections but also require PMTs to actively employ this integrated knowledge in simulated or authentic teaching contexts, thereby bridging the gap between their understanding and their capacity to explain mathematical reasoning to others. It underscores, as well, the necessity of sustained, multi-faceted interventions rather than one-off experiences. Future designs must incorporate repeated and varied opportunities for PMTs to apply structural reasoning across different mathematical domains.

Indeed, a parallel development in George's MKT of could also be observed. His pedagogical performance throughout the implementation demonstrated extensive growth, particularly within the domains of SCK and HCK. This was evidenced by his intentional and skilful orchestration of classroom discussions to explicitly highlight the structural properties underlying mathematical procedures, moving beyond a purely procedural teaching approach. Furthermore, his capacity to be reflective, as captured in the post-implementation interview, indicates a maturation of his KCT. George demonstrated an enhanced ability to critically evaluate his instructional choices, specifically the shift from an axiomatic presentation of number sets to a constructive one, that considers the implications for future teachers' understanding. This critical reflection on the how and why of teaching specific content is a cornerstone of advanced KCT.

Finally, George's strategic decision to employ specific discursive moves—such as presenting simplified analogous situations to guide PMTs toward self-correction rather than providing direct answers—serves as a powerful instance of modelled instruction. This not only fostered a deeper conceptual understanding among the PMTs but also explicitly demonstrated a high-leverage teaching practice, thereby contributing to the PMTs' own developing KCT. Thus, the intervention served as a powerful professional development experience for the educator, catalysing an integrated development of their MKT.

The case of George is particularly illustrative. His journey exemplifies the potential for significant professional growth. George's post-implementation reflection indicates a shift in his beliefs. George moved from implicitly assuming connections existed to consciously and meticulously making them explicit, dedicating substantial class time to deconstructing conceptual errors and their implications for secondary teaching in a subsequent class, following the implementation of a topic distinct from that of the researcher-led intervention (arccosine). This evolution suggests development not only in his SCK, HCK and KCT but also in what might be termed MTE's pedagogical imperative—a belief in the responsibility of an MTE to bridge these mathematical worlds for their students.

Final Remarks

In conclusion, this study provides empirical support for the following claims:

For Mathematics Teacher Educators:

Engaging in collaborative design and implementation of sequences that explicitly connect advanced and secondary mathematics acts as a powerful catalyst for their own MKT development. It fosters a reflective stance towards their subject matter and its pedagogical implications, transforming their teaching practice from the delivery of abstract content to the modelling of mathematical connections.

For Prospective Mathematics Teachers:

Such intentionally designed sequences can effectively mobilise their inert knowledge of advanced mathematics, helping them to develop a properties-based, structural view of school mathematics. This is a fundamental component of robust SCK and HCK.

For Teacher Education:

The "pedagogical content knowledge" of an MTE must include the specific skill of unpacking and articulating the connections between the advanced mathematics they teach and the school mathematics their students will teach. This skill can be effectively developed through structured professional activities like the workshops described herein.

Future Research

The primary limitation of this study is its small scale and focus on a single, in-depth case. While this provides rich qualitative evidence, broader generalisability requires further research with a larger cohort of MTEs and PMTs. Future work should focus on several areas. Longitudinal studies could track the persistence of the shifts in MTEs practice and their long-term impact on PMTs. Larger-scale implementations could examine the efficacy of the proposed theoretical model on relationships between types of connections and MKT (Figure 1) across different institutional contexts. Further research could also investigate the transfer problem by designing and studying interventions that support PMTs in applying structural reasoning consistently across numerical, algebraic, and geometric contexts. Ultimately, this research argues that the development of MKT is not a solitary pursuit of the prospective teacher but a collaborative endeavour that must be modelled and nurtured by teacher educators and researchers, working in collaborative ways.

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Ethical approval

Ethical approval for the research was granted by the Council for Teacher Education (CFE, Uruguay), and informed consent was given by all participants for their data to be published. Pseudonym was used when referring to MTE George.

Competing interests

The authors declare there are no competing interests.

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