Using Reasoning-and-proving Cycles to Foster Pre-service Teacher Development in Argumentation and Proof: An Illustration of Practice

Gabriel Chun-Yeung Lee
University of Nottingham

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This illustration of practice features two reasoning-and-proving tasks that fostered pre-service teachers' development in argumentation and proof. Argumentation and proof are significant to the learning of mathematics. In a design-based study, a teacher development programme was developed and tested through multiple research cycles. The programme was shown to be effective in promoting pre-service teachers' positive beliefs and knowledge about argumentation and proof. In the programme, the two reasoning-and-proving tasks linked argumentation and proof to the work of teaching. Each task featured one of the two key topics in school mathematics—properties of integers, and circles. The tasks modelled the process of reasoning-and-proving and invited learners to explore the featured mathematical ideas through multiple methods, representations, and cycles. The tasks provided pre-service teachers with alternative experiences in argumentation and proof, arousing positive emotions and motivation. This illustration of practice also discusses the key role that teacher educators could play in using the tasks to support teacher development.

Keywords • argumentation and proof • circles • consecutive numbers • reasoning • proving • tasks for teacher development

Introduction

Argumentation and proof are significant to the learning of mathematics (Hanna & Barbeau, 2008; Stylianides et al., 2016; Zaslavsky et al., 2012). However, it is hard for teachers to teach argumentation and proof. Many researchers have attempted to identify and address issues around the teaching of proof. For example, teachers do not usually have sufficient knowledge or productive beliefs to motivate themselves to include argumentation and proof in their classrooms (Stylianides et al., 2023; Stylianides et al., 2017). Even if they have, pre-service teachers often have limited previous experience of argumentation and proof to support their teaching meaningfully (Buchbinder & McCrone, 2020; Stylianides et al., 2013). Interventions have also been developed to foster teachers' professional development in varying aspects of argumentation and proof (e.g., Buchbinder & McCrone, 2020, 2023; Stylianides & Stylianides, 2009). However, more studies and examples are needed to support teacher educators' work of searching for concrete proof-related tasks that meet their pedagogical needs. Drawing upon research on close relationships between teacher beliefs, emotions and practice, I developed a teacher development programme and reported on its effectiveness in supporting pre-service teachers' positive beliefs and emotions about argumentation and proof in a design-based study (Lee, 2026). In this illustration of practice, I discuss two featured tasks used in the programme that fostered participants' development in argumentation and proof. To make this illustration of practice more accessible, the list on the next page shows the meanings of the proof-related terminology used in this illustration of practice.

Conjecture: a hypothesis about a claim that one thinks is true, based on incomplete evidence. e.g., "*I think* the sum of two consecutive numbers is always odd". Proposition: a claim that can be true or false. e.g., "the sum of two consecutive numbers is always even" (indeed, this is false). Generalisation: a product/process of transporting a claim from given set(s) to more general set(s). e.g., a *finite* set of cases that meets the true-ness of a proposition. "1 + 2 = 3" is an example of "the sum of two consecutive numbers is odd." Argument: a product that connects claims from premises to conclusion. Proof: a valid argument based on logical rules and principles and accepted true claims for or against a proposition. Argumentation: a process of making claims in the pursuit of an argument or a proof.

The two tasks highlighted the reasoning-and-proving framework (G. J. Stylianides, 2008). Whilst this framework was initially designed for analytical use (e.g., analysis of textbooks), this illustration of practice contributes to the literature by using the framework for instructional design. As a design framework, it was extended to incorporate a cyclic feature in which a mathematical generalisation, once justified by valid arguments, can be re-explored through different modes of representation and/or further generalised to a less restricted proposition. Within this cyclic framework, a task is not merely seen as a single "proving" task in which learners identify a pattern, formulate and justify a conjecture. Instead, learners pay attention to details of the argument(s) underpinning a generalisation, revisiting and extending their reasoning to deepen their mathematical understanding through inquiry (Skovsmose & Säljö, 2008). This illustration of practice presents two tasks that adhere to this cyclic framework, and examples of teachers' work on the tasks, together with a discussion on the practicality of the tasks in teacher development.

Reasoning-and-proving Tasks

Two reasoning-and-proving tasks were initially designed for teacher development in argumentation and proof in a design-based study (Lee, 2026). The tasks lasted 45 minutes each and featured two prominent mathematical components of the reasoning-and-proving framework—making mathematical generalisations and providing arguments (G. J. Stylianides, 2008). In the study, the tasks were refined and tested through multiple design cycles, with multiple groups of pre-service teachers in primary and secondary mathematics. In the development stage, the tasks were also used with secondary schoolteachers—it has been shown that the tasks are relevant to the actual work of teaching school mathematics (Lee, 2026). The first task aimed to explore divisibility of sum of consecutive numbers, and the second task aimed to arrive at area of a circle, both relevant to the learning and teaching of school mathematics, in the international context (e.g., Australia: Australian Curriculum, Assessment and Reporting Authority, 2022; Lehmann, 2025) and in the context of the study—Hong Kong (Education Bureau, 2020a, 2020b). Both tasks started with a warm-up activity in which mathematical words relevant to the topic were explored. Through instructor's prompting, the participants identified patterns and produced arguments to support their claims about the patterns. The tasks also required the participants to reflect on the ideas of examples, arguments and proof, from both mathematical and pedagogical perspectives (G. J. Stylianides, 2008). Doing so, the tasks connected the topics and argumentation and proof, and engaged the participants as learners of mathematics and pedagogy, thereby fostering professional development (Goos & Geiger, 2010; Shilling-Traina & Stylianides, 2013). As the tasks were situated in school mathematics, different modes of representation (e.g., drawings) could be accepted in an argument (A. J. Stylianides, 2007).

Task 1: Consecutive Numbers

The Consecutive Numbers task followed an activity in which participants discussed videos of a classroom instruction on odd and even numbers (Mathematics Teaching and Learning to Teach, 2010a, 2010b).

The participants realised different representations of numbers; for example, an odd number can be represented algebraically by $2n + 1$, and pictorially by two rows of dots with the second row having one extra dot. This task started with a question "What is the divisor of the sum of three consecutive integers?" Some participants first tested different number combinations and identified that "sum of any three consecutive numbers can be divided by 3" (the proposition). Some participants appeared to choose their number combinations structurally (left, Figure 1), whereas some picked their examples rather arbitrarily (right, Figure 1). At this stage, the examples appeared to serve merely for making a generalisation, but not yet for understanding why (Ozgur et al., 2019).

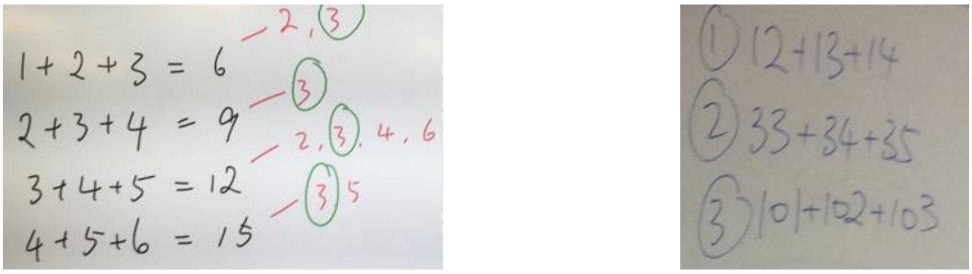

Figure 1. Empirical examples of sum of three consecutive numbers.

Some other participants used the same set of numeric examples differently or used pictorial representations of consecutive numbers (rows of dots) to examine the proposition (Figure 2). Using these examples, the participants gained an insight into the proposition: the sum of three consecutive numbers is divisible by 3, because the first (smallest) and the last (largest) numbers "complement" each other to make "two" second numbers; together with the original second number, there are "three" second numbers altogether, thereby making the sum a multiple of 3. Note that the participants did not rely on only one mode of representation. Unlike the examples in Figure 1, the participants used the examples in Figure 2 to convey a general argument (Ozgur et al., 2019), as shown in the ellipses (…) used. During the study, ellipses were used in two ways: (a) to omit a repeated pattern (left, Figure 2), and (b) to represent an unknown quantity (right, Figure 2).

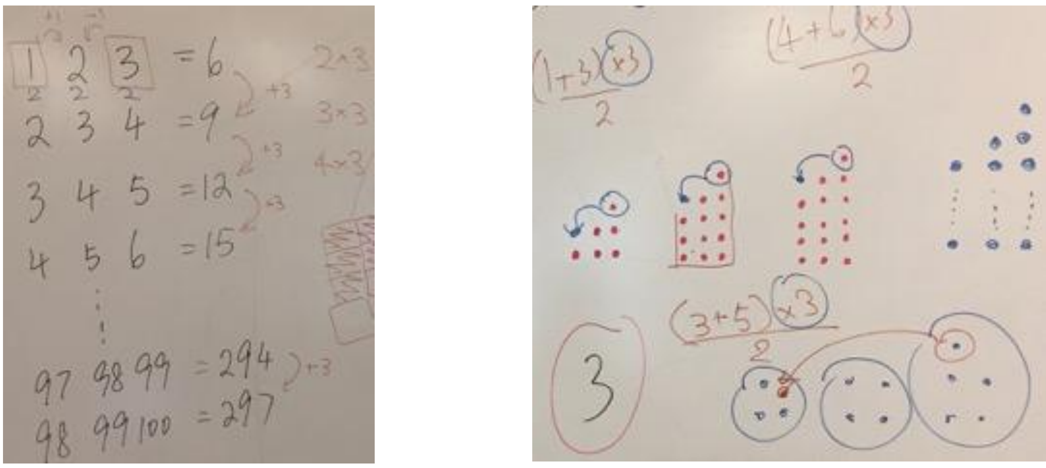

Figure 2. Generic examples of sum of three consecutive numbers.

The use of multiple representations helped the participants connect and articulate mathematical ideas. Some participants identified the relationships between consecutive numbers (i.e., any paired consecutive numbers are differed by 1) and developed their argument for the proposition from numerals (generic examples) into algebraic expressions (left, Figure 3). Similarly, some participants developed their argument from pictorial representations (right, Figure 3). As such, examples in different representations were used to support understanding and development of justification of the proposition (Ozgur et al., 2019). Indeed, not all participants developed a proof for the proposition in the first instance, but they were prompted to share their initial thoughts about the proposition and the examples. The participants discussed and made connections between their examples. This exchange of ideas fostered active learning and resulted in the development of proofs for the proposition.

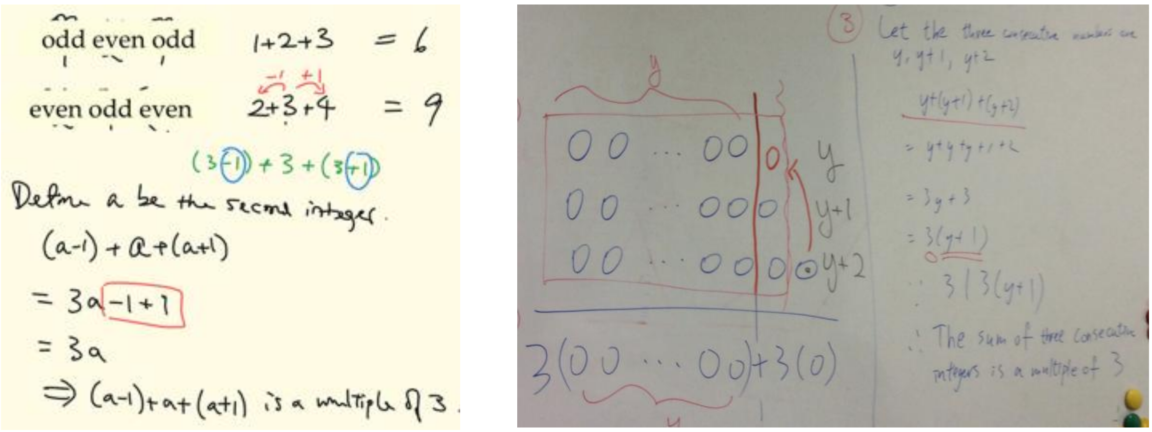

Figure 3. Proofs of sum of three consecutive numbers.

After the participants were satisfied with their results—the proposition "the sum of three consecutive numbers is divisible by 3" and its proofs, they were prompted to further generalise the proposition (Figure 4). Some participants immediately found that the new proposition, "The sum of *n* consecutive numbers is a multiple of *n*" was not true, as the example of four consecutive numbers "1 2 3 4" and the sum "10" refute the proposition when $n = 4$. However, the task did not stop there: the participants continued to identify examples and patterns that fitted the proposition. They discovered that the proposition holds true when *n* is odd. Some participants built on their previous arguments for the proposition when $n = 3$ and produced a generic example: when *n* is odd, the numbers that are greater than the middle number complement the numbers that are less than the middle number, resulting in *n* times of the middle number (right, Figure 4). At this stage, the participants had engaged with examples and non-examples of the proposition that served different purposes from understanding the proposition, refuting a conjecture, exploring the truth domain of the proposition, to conveying and communicating an argument (Ozgur et al., 2019).

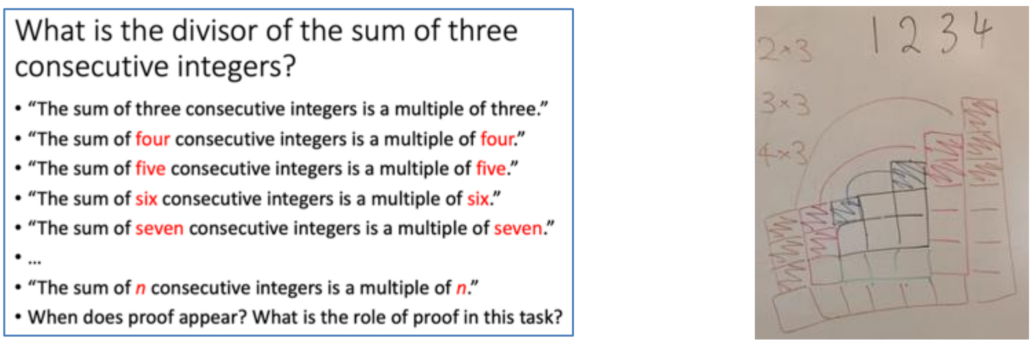

Figure 4. Task of sum of consecutive numbers.

The participants actively talked about their mathematical work. Through instructor's prompting, the participants produced multiple solutions to the task and refined their work. When realising that merely presenting empirical examples would not defend the proposition of this task, the participants shifted attention from producing more examples to identifying key ideas within the examples that convey a general argument. Later, the prompting and the need for multiple solutions also helped the participants to engage in a pedagogical discussion on argumentation and proof and how examples supported their argument development.

Task 2: Circle

The Circle task was the closing task of the teacher development programme. The task began with a question "What do you know about circles?" The question served to direct the participants' attention to their existing knowledge about circles, at school level. "Centre", "radius" and "infinitely many axes of symmetry" are examples of what most participants recalled from the senior primary and junior secondary mathematics curricula in Hong Kong (Education Bureau, 2020a, 2020b). The question and the participants' responses set up the context and prepared them for the next activity:

Each participant was provided with a pair of scissors and several pieces of square paper (e.g., 10 cm × 10 cm origami paper) and asked to cut a "circle" as big as they could. After that, the participants presented their "circles" and explained how they made the circles. Figure 5 shows a summary of different methods that the participants used. For example, some participants used tools, such as a compass, a coin, or a bottle, to outline a circle before cutting, and some outlined a circle with bare hand. Attention was paid to the idea that participants folded the paper several times to produce axes of symmetry. Some participants identified a pattern that linked the number of axes of symmetry (the number of times of paper being folded), the figure produced (e.g., a polygon), and circle: when the number of axes of symmetry of a (regular) polygon increases, the polygon gets close to a circle.

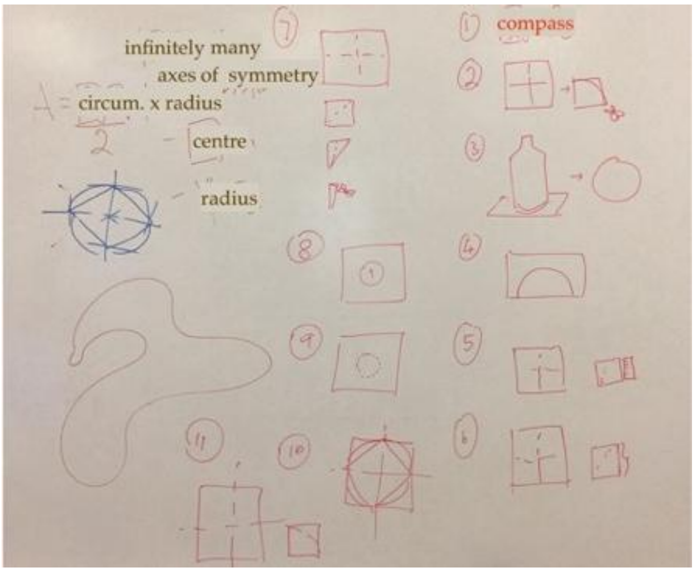

Figure 5. Different methods of cutting a circle.

This inspired the participants to revisit the formula of the area of a circle, which they memorised and used for solving mathematical tasks, but only a few participants could explain why the formula is true. The pattern helped the participants "re-construct" one commonly used argument in school mathematics: in which a circle is dissected and rearranged to form a figure close to a parallelogram (left, Figure 6). As the number of dissected sectors increases, the rearranged "parallelogram" gets closer to a rectangle. Since the length and height of the "rectangle" approximate half of the circumference and the radius of the circle, respectively (right, Figure 6), the area of the circle becomes: ½ × *circumference* × *radius*. As the method involves "thin triangles" for approximation, participants also discussed the inappropriateness of replacing the triangles with "thin rectangular strips" for the formula (right, Figure 6): the strips overlap around the centre, overestimating the area.

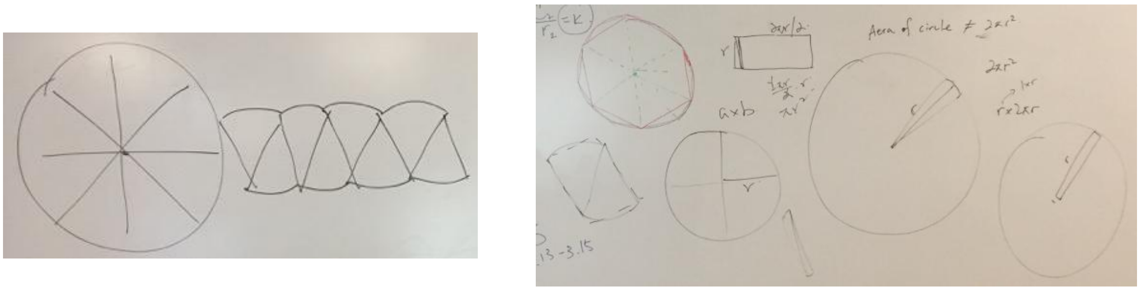

Figure 6. Method of dissecting a circle into sectors.

Some participants also produced another argument for the same formula, by creating concentric rings from the circle: the rings approximate to a series of rectangular strips, which are then rearranged into an "approximate triangle" (left, Figure 7). Since the approximate triangle has a base length equal to the circumference of the circle ($2\pi r$) and its height is the radius of the circle (r), the area of the "triangle" approximates the area of the circle: ½ × *circumference* × *radius* = $\frac{1}{2} \times 2\pi r \times r = \pi r^2$ (right-bottom, Figure 7). Participants were also prompted to discuss the shape of the approximate triangle (i.e., the lengths of the rectangular strips form a linear pattern; right-top, Figure 7), which is linked to the claim that "the length of a strip is proportionate to the distance between the strip and the centre of the circle" (see next). The two methods involve the concepts of approximation and limits (Dreyfus, 1987) and can benefit school students in argumentation (Lehmann, 2025; Stacey & Vincent, 2009).

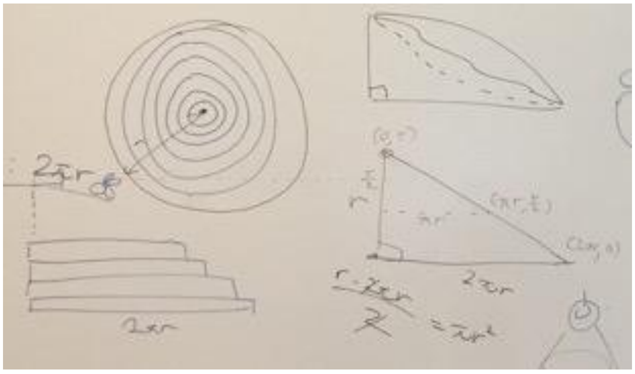

Figure 7. Method of dissecting a circle into concentric rings.

This task did not stop after the two methods were produced and discussed from pedagogical perspectives; the reasoning-and-proving cycle was extended into two different strands. The first strand focused on the concept of π as a mathematical constant, whereas the second strand searched for a third argument for the formula of the area of a circle. In this study, participants automatically linked π to the circumference (C) of a circle: $C = 2\pi r$. And when asked about the concept of π, the participants referred to an argument for π as the constant ratio of the circumference of a circle to its diameter ($2r$) by measuring different circles and plotting a linear graph of the empirical data (left, Figure 8). However, this empirical argument could not account for the constant nature of π (Leung, 2014). Some participants then suggested that it could be argued by the fact that all circles are *similar* to each other (right, Figure 8). The concept of similarity enabled the participants to answer the question about the shape of the approximate triangle in the method of concentric rings (Figure 7) and realise that π, as the ratio of the circumference of a circle to its diameter ($C/2r$), is independent of the size of the circle (Leung, 2014). This required the participants to connect geometric property (i.e., similarity of circles) to algebra (i.e., formulation of an equation that shows the equivalence of the ratios of the circumferences and diameter/radius of any two circles) (Leung, 2014). This result supplemented the algebraic manipulation that the participants used to reach the formula of the area of a circle.

Figure 8. The constant nature of π.

A few participants attempted to produce a third argument for the formula of the area of a circle, based on the conversation in the paper-cutting activity about the relationship between a circle and its inscribed *n*-sided polygon. This argument differed from the previous arguments, as it did not involve dissecting and rearranging a figure but focused more on algebraic manipulation of labelled expressions. The area of the polygon can be seen as a function of its perimeter and apothem (Figure 9). As the number of its sides increases, the polygon gets closer to a circle: (a) its area approximates the area of the circle, (b) its perimeter approximates the circumference of the circle, and (c) its apothem (*h*) approximates the radius of the circle. In Figure 8, consider the regular n-sided polygon inscribed in a circle; as *n* increases,

$\angle CAD = ½ \times \angle CAB = ½ \times (360°/n)$ tends to 0. Therefore, $h = AD = \cos\angle CAD \times AB$ tends to $1 \times AB$ (i.e., radius of the circle).

In this sense, the area of a circle is therefore computed by: ½ × *circumference* × *radius*. This argument drew on a dynamic geometry resource (https://www.geogebra.org/m/y5dfm7qb). This resource illustrated that a regular polygon gets closer to a circle when the number of its sides increases (Figure 9). One may deem that this argument extends beyond junior secondary mathematics, due to the involvement of a cosine of an angle and its property (i.e., cos θ tends to 1 as θ tends to 90°) and the algebraic manipulation involved. However, this argument is meaningful and fruitful as it is nicely built on the paper-cutting activity, and more importantly, this argument was produced by the participants using and connecting various areas of mathematics. The process of connecting multiple areas of mathematics enabled participants to deepen their understanding of the topics of interest, fostering their professional development.

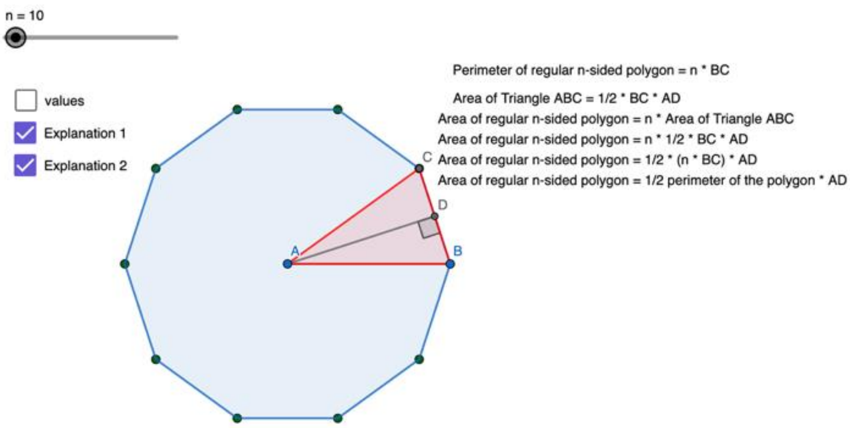

Figure 9. The relationship between the area of a regular *n*-sided polygon and its perimeter.

Outcomes and Discussion

In the previous section, two reasoning-and-proving tasks and examples of participants' work were presented to illustrate how the tasks can support teacher development in argumentation and proof. During the tasks, there were "aha!" moments observed or reported by the participants afterwards. The tasks provided the participants with experiences that differed from their prior experiences in those topics and in argumentation and proof. These alternative experiences arouse the participants' positive emotions and motivation in argumentation and proof (Lee, 2026). By fully engaging in the tasks as learners of mathematics, the participants not only developed more connected, sophisticated understanding of the topics, but also reflected on their pedagogies:

> Participant A: I've learnt [...] when learning a proof students can first try to explore [...] some simple things to [learn] what the proof is about. Then, let them write the statement—the conjecture, and let them try to prove it [...] It's fine for them to experience this [reasoning-and-proving] process [...] After I attended this workshop, I've realised many different topics and many different strategies [for] teaching.

It is worth noting that presenting the tasks does not automatically translate into learning and practice development. It is the role of the instructor to maintain the meaningful experiences when participants engage in the tasks (Lee, 2026). To help participants engage fully in the tasks, teacher educators, as the instructor, could pay attention to questioning. As illustrated in the previous examples, prompting enables participants to reflect on their work and explore different solutions, methods, and representations, creating meaningful challenges. When these challenges are achievable and achieved by participants, positive emotions, such as joy, interest, contentment, and pride, would be aroused (Csikszentmihalyi, 1990; Fredrickson, 2001), resulting in learning and development.

> Participant B: I think, during the workshop, [it was] like having a storm in [my] brain. Many unexpected questions popped out [...] so I find [them] interesting, which looked like simple but not so easy to comprehend.

The tasks developed contribute to the field of teacher education as they demonstrate how the reasoning-and-proving framework can be used for instructional design. Examples of participants' work and reflections on the tasks provide insights into their development in argumentation and proof during the tasks and into practicality of the tasks. The tasks form the basis for challenges, whereas instructor's prompting serves to extend the level of difficulty when participants achieve a target. The cyclic nature enriches the tasks and deepens participants' mathematical understanding. A task does not necessarily end when a problem is solved or a proof is produced, but it can be extended by enquiring into a relevant proposition. When participants have difficulties with a task, prompting can serve to draw participants' attention to information that can help achieve the target or help to break down a challenge into achievable steps. Instructor's expertise, therefore, plays a role in supporting participants' development.

Corresponding author

Gabriel Chun-Yeung Lee
Observatory for Mathematical Education
University of Nottingham, Nottingham, United Kingdom NG8 1BB
gabriel.lee@nottingham.ac.uk

Ethical approval

Ethical approval was granted by the University of Oxford Central University Research Ethics Committee (Ref: ED-CIA-18-198) and the Education University of Hong Kong Human Research Ethics Committee

(Ref: 2018-2019-0323) for the data to be reported. The participants gave informed consent for the use of their data in the study for publication purposes.

Competing interests

The authors declare there are no competing interests.

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