The Shift From "Learner/Doer of Mathematics" to "Teacher of Mathematics": A Heuristic for Teacher Candidates

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Successful teacher preparation programs provide learning experiences that help teacher candidates make the shift from "student" to "teacher." In this paper we present research on the implementation of a process for providing candidates such experiences. Utilizing the Mathematics as Teacher Heuristic (MATH) process, prospective high school mathematics teachers explore rich problems by solving the task, analyzing samples of student work, designing a solution key, and modifying the task. We use their engagement in these explorations and reflections on the process to analyse the development of candidates' Content Knowledge and Pedagogical Content Knowledge.

Keywords: preservice teachers • teacher knowledge • mathematical knowledge for teaching (MKT) • teacher enculturation • student work

Mathematics as Teacher Heuristic (MATH)

Successful teacher preparation programs provide learning experiences that scaffold a path for teacher candidates in making the shift from "student" to "teacher." This transition is a crucial goal: to be a successful teacher a candidate must learn to see the tasks of school mathematics from another perspective. However, this is not an easy transition, particularly for future school mathematics teachers; complicated, as it often is, by the fact that candidates' interest in mathematics and self-efficacy regarding mathematics arises from their personal success in solving complex problems rather than in their ability to understand others' solutions.

In the course of attempting to scaffold this transition for teacher candidates and reflecting on what we have found to be helpful, a five-step approach has evolved whereby we feel we can facilitate this transition effectively for teacher candidates in the context of examining rich content tasks. The approach, which we refer to as the Mathematics as Teacher Heuristic (MATH), is designed to provide a set of experiences that gradually require candidates to shift their mathematical view from a "learner of mathematics" orientation to one embracing teacher-oriented perspectives. The MATH process requires candidates to engage with rich mathematics tasks, but from points of view rarely considered in content-oriented coursework which are concerned with candidates' attempts to do their own mathematics rather than consider another learners' mathematics.

In the MATH procedure, candidates complete a five-step process that includes:

1. solving a rich task as a "doer";
2. assessing student work samples associated with the same task;
3. constructing "classroom ready" solution keys for students;
4. developing scaffolded instructional materials addressing student challenges, difficulties, and misconceptions (gleaned from earlier analyses); and
5. reflecting on the process.

Each step of the MATH process encourages candidates to consider the rich tasks from increasingly teacher-centric points of view. In this sense the process can be seen as a heuristic since it provides a specific technique that can be applied repeatedly to scaffold the necessary shift in perspective a teacher candidate must undergo. Assessing authentic student work and constructing possible solution keys are activities that require candidates to interpret a learner's work and consider guidance for a learner (teacher-oriented tasks) rather than solving the problem on their own terms and presenting it for consideration (a learner's perspective).

Ultimately, the instructional materials that candidates create as part of the MATH process are shared with practising teachers and their students. By creating work for audiences beyond the university classroom, candidates find the work more meaningful. As they construct solutions and materials for classroom teachers and their students, candidates recognise that "finding an answer" is no longer the central purpose of their mathematical work. Rather, problem solving is situated in the context of helping students learn mathematics—a teacher-oriented behavior rather than a student-oriented one. As candidates construct learning materials, they discuss mathematics content in the methods classroom from a learner's viewpoint. The approach provides a useful basis to analyze the development of candidates' Pedagogical Content Knowledge (PCK) (Shulman, 1986): a combination of Content Knowledge (CK) and Pedagogical Knowledge (PK).

In this paper we analyse the work of one group of candidate teachers as they explored a rich mathematics task using the MATH process. In particular, we examine candidates' analyses of student work, their solution keys, their modifications of the task, and their reflections and insights on the MATH process as a whole.

Related Literature

Many models for understanding the transition from "student" to "teacher" focus on dissonance and motivation in school settings, factors that occur regularly for in-service teachers in their day-to-day practice (Clarke & Hollingsworth, 2002; Edwards, 1994). Clarke and Hollingsworth (2002) and Loughran (2002) stress the importance of self-reflection in the development and evolution of teacher knowledge, beliefs, and attitudes. We believe that candidate teachers need to experience similar dissonances in university coursework in order to accept the need for change and reflect on their change process. Providing such opportunities for change and reflection is perhaps more difficult in the case of candidate teachers at the very beginning of their course of study since they have fewer authentic teaching experiences to draw upon. Studies show that two main sources for dissonance in initiating candidate growth are in methods courses and student teaching.

Brown and Borko (1992) argue that there are three important issues in the process of "learning to teach"; namely (1) the influence of content knowledge, (2) novices' learning of pedagogical content knowledge, and (3) difficulties in acquiring pedagogical reasoning skills. Furthermore, they assert that

> … one of the most difficult aspects of learning to teach is making the transition from a personal orientation to a discipline to thinking about how to organize and represent the content of that discipline to facilitate student understanding (p. 221).

In a case study of one teacher's successful transition from student to teacher, Velez-Rendon (2006) identified a complex interplay between many factors during this transition including "the learning background the participant brought with her added to her knowledge of the subject matter" (p. 320). Similar to Brown and Borko's (1992) results, Velez-Rendon concluded that the preparation of the subject matter for instruction was challenging for the candidates.

Another related issue is the "tension between participants' views of themselves as adult learners of mathematics and their practice with young children" (Brown & Borko, 1992, p. 215). Ball (1989) studied the role of a methods course for elementary mathematics teachers in helping candidates to learn to teach. Ball highlights the importance of content knowledge and the experiences of candidate teachers as learners of mathematics.

> Unless mathematics teacher educators are satisfied with what prospective teachers have learned from their experiences as students in math classrooms (and most are not), this highlights a need to interrupt, to break in, what is otherwise a smooth continuity from student to teacher (Ball, 1989, p. 4).

In this sense, we see Ball's suggestions to break with experience as an opportunity to create a dissonance.

A further crucial aspect of the transition from doer to teacher of mathematics is examining the nature of their content knowledge and developing mathematical knowledge related to teaching. Research led by Bass (2005), Ball, Hill and Bass (2005) and Hill, Rowan and Ball (2005) has provided "an emerging theory of what we have named *mathematical knowledge for teaching* (MKT)" (Bass, 2005, p. 423). Bass characterises this knowledge as "the mathematical knowledge, skills, habits of mind, and sensibilities that are entailed" in teaching mathematics (p. 429).

Hill (2007) details some of what is included in MKT:

> ... mathematical explanations for common rules or procedures, common non-symbolic representations (or links between representations) of mathematical subject matter, the ability to unpack and understand non-standard solution

> methods, and using mathematical definitions in accurate yet also grade-appropriate ways. This knowledge, like common content knowledge, is wholly mathematical … However, it is not mathematical knowledge that any nonteaching adult would necessarily possess. (p. 98)

There is evidence that development of MKT is directly correlated to student achievement (Hill & Lubienski, 2007; Hill, Rowan & Ball, 2005) and a growing consensus that such knowledge is important for mathematics teachers (Ball, Thames, & Phelps, 2008). Rowland, Huckstep and Thwaites (2005) draw on the construct of MKT and how it relates to teaching of mathematics to develop a "knowledge quartet" of "foundation, transformation, connection, contingency" (p. 259). They describe possible stages of a teacher candidate shifting their perspective from their own knowledge, to mathematics, to the concerns and perspectives they must have as a teacher of mathematics.

Silverman and Thompson (2008) further extend the notion of MKT and examine the construct specifically as it relates to mathematics educators. They draw on Simon's (2006) notion of a key developmental understanding (KDU), defined as "change in the learner's ability to think about and/or perceive particular mathematical relationships" (Simon, 2006, p. 362), and consider how to extend this notion to educators as opposed to learners. As Silverman and Thompson say "Teachers who develop KDUs of particular mathematical ideas can do impressive mathematics with regard to those ideas, but it is not necessarily true that their understandings are powerful pedagogically" (p. 502). Silverman and Thompson develop a framework for the goals of mathematics teacher preparation which uses candidates' well-developed KDUs as a first step of a model of knowledge which supports teaching—the second and third steps of which are construction of "models of the variety of ways students may understand the content (decentering) [and having] an image of how someone else might come to think of the mathematical idea in a similar way" (p. 508). In other words, teacher candidates must shift (be decentred) from an understanding of mathematics for themselves to thinking about how someone else understands, engages in, and might be taught mathematics.

In another effort to describe and measure the shift undertaken by teacher candidates; Hill, Ball and Schilling (2008) developed the construct of "knowledge of content and students (KCS)" (p. 373), a combination of subject content knowledge and knowledge of how students think about and engage with content. Such knowledge directly involves prospective teachers making a shift from thinking about mathematics content in terms of their own knowledge to thinking about students, students' knowledge, their misconceptions, and typical practices for engaging students with particular mathematics content. Examining the nature of candidates' MKT, KCS, and KDUs provides another challenge and opportunity for dissonance for teacher candidates as they discover that their own ability to solve a particular problem is only one part of the mathematical knowledge they need to be an effective teacher.

Related to Silverman and Thompson's (2008) "decentering" and of particular relevance to the kind of shift we are asking teachers to make is the notion of "unpacking", as discussed in the work of Adler and Davis (2006). In one of their examples of unpacking, they present five different student responses to a standard question requiring finding solution(s) of a quadratic and note that, after seeing at a first level of analysis that all the students have found a correct answer, "The teacher will need to unpack the relationship between a mathematical result or answer and the process of its production" (p. 274). Adler and Davis note that the teacher is also faced with the challenge of interpreting the specific strategies used by each student and consider how those strategies, some of which are incomplete or problematic, will be orchestrated in a classroom setting to consolidate the learning of all students. Engaging in such understandings which are part of the MKT construct involves the creation of a dissonance whereby teacher candidates are challenged to think about mathematics in ways that are not their own and displace them from the role of "doer" of mathematics.

The importance of the shift from "doer" to "teacher" has been well established in the literature. Thompson (1984) has demonstrated that "teachers' beliefs, views, and preferences about mathematics and its teaching played a significant, albeit subtle, role in shaping their instructional behavior," (p. 105). Further research by Pajares (1992), and Leder, Pehkonen, and Torner (2002) has highlighted the challenging and confounding influence of the beliefs about mathematics that teacher candidates bring to their training, and Cooney (1999) refers to their privileging of more abstract algebraic solutions.

Loughran (2002) explored the development of knowledge through effective reflective experience from teacher candidate to experienced teacher, comparing candidates' views of teaching and learning with those of practitioners. Candidates' views regarding teaching and learning typically equated learning "with gaining right answers" (p. 41). This naive view of teaching contrasts markedly with teacher comments that emphasise the importance of "student … opportunities to be active and think about their learning experiences" (p. 41). Loughran's study illustrates the importance of giving candidates opportunities to face their views, reflect, and reconsider them. Swafford, Jones, Thornton, Stump, and Miller (1999) echo these findings for inservice teachers. Swafford et al. recommend the creation of environments for teachers that improve their content and pedagogical knowledge through reflection and collaboration.

Moving from the back of the table to the front of the blackboard is not a smooth or simple journey (see Figure 1). Candidate teachers carry the baggage of their previous experiences obtained as students. Those experiences affect not only their journey and their content knowledge but also their beliefs, views, attitudes, and values. A candidate's affective domain influences how they utilise their own knowledge (content knowledge, pedagogical knowledge, and pedagogical content knowledge) in classrooms and in their journey. How, then, can we can help our teacher candidates to distance themselves from their student identity during their methods courses? We believe that a partial answer exists in reflective problem solving activities embedded within the MATH process.

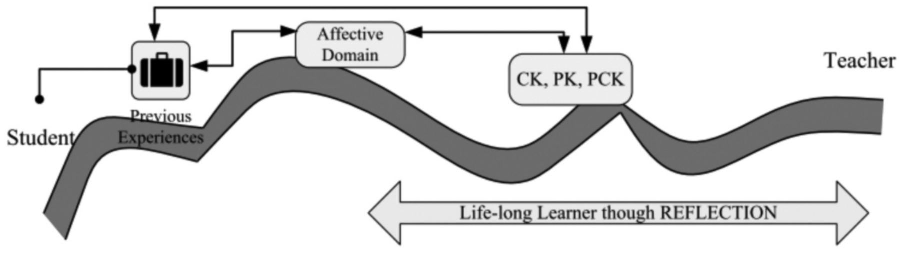

Figure 1. Student to teacher journey.

Student versus Teacher Problem Solving

How to engage individuals with little or no formal teaching experience in activities that encourage thoughtful examination of school mathematics pedagogy and content *as teachers* is a dilemma familiar to those who work with teacher candidates in methods courses. Differences between student problem solving and teacher problem solving are depicted graphically in Figure 2.

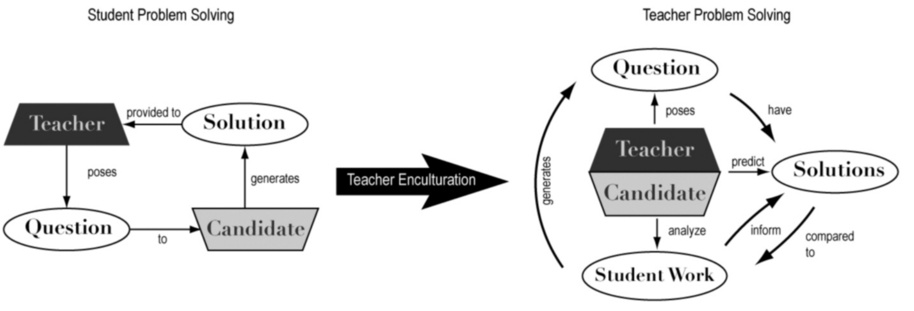

Figure 2. Student problem solving versus teacher problem solving.

In Figure 2, the arrow labeled "teacher enculturation" represents the gradual process by which candidates learn to approach problems as teachers. This enculturation process ideally begins as candidates enter teacher education programs and begin to work with children in field placements. Our notion of enculturation is influenced by Bishop (1991), although our work focuses on the enculturation of teachers rather than of school students.

The portion of Figure 2 to the left of the "Teacher Enculturation" arrow (labeled above as "Student Problem Solving") depicts a naive view of the problem solving process held by many pre-service teachers prior to enculturation (i.e., in the early stages of methods instruction). In this portion of the Figure 2, teacher and candidate are seen as separate entities. The methods teacher is seen as the lone source of mathematics tasks—tasks that are solved exclusively by teacher candidates. On the other hand, the portion of Figure 2 to the right of the enculturation arrow (labeled above as "Teacher Problem Solving") represents a more mature sense of the problem solving process—one that we strive to engender in teacher candidates through their experiences throughout methods coursework. In this portion of the model, teacher and candidate are seen as a single group (labeled Teacher/Candidate). Teacher candidates pose questions and develop tasks in the methods classroom. Next, they predict how school students may solve these tasks. After tasks are posed in authentic settings to students in school classrooms, teacher candidates have opportunities to compare their predictions to actual classroom outcomes. Analysis of these differences informs the revision of the posed tasks as well as the construction of new items in a cycle of problem construction, posing, and analysis. While experiences in the field may provide candidates with a *glimpse* into the dynamic world of school classrooms, too often such episodes are too brief to provide future teachers with a true sense of the complexities of day-to-day teaching.

Moreover, a disconnection often exists between the best practices promoted by methods instructors in university classrooms and the teaching that candidates observe in field placements. In an age fraught with ever-expanding curricular demands fueled by high-stakes testing, securing venues for field placements is problematic at best. Experienced master teachers—those committed to providing their students with high-quality instruction and those best equipped to model such instruction to teacher candidates —are often reluctant to forfeit precious instructional time to share their craft with the next generation of educators.

Methodology

Participants in this study were candidates enrolled in EDT II, the second course in a year-long methods sequence designed for prospective secondary mathematics teachers. Twenty-three of the 29 teacher candidates were undergraduates; six were graduate students who had returned to university seeking teacher licensure in secondary mathematics education. In the previous methods course (EDT I), candidates had spent significant time solving rich mathematics tasks as students. The intention of problem solving experiences in the first semester was two-fold; namely (1) to reintroduce the content of secondary school mathematics to candidates; and (2) to communicate to candidates the notion of "rich" mathematical tasks.

In the follow-up course (EDT II), we revisit a subset of these rich tasks with teacher candidates using the five-step MATH process. In the paragraphs that follow, we explore data generated through the exploration of one such task, namely the Bridgewater Problem.

Candidate Tasks

Candidates solved the following task in the first semester methods course (see Figure 3).

The Bridgewater Problem. At 1:00 p.m., two hikers began walking, the first from Amityville to Bridgewater, the second from Bridgewater to Amityville along the same path. Each walked at a constant speed. They met at 4:00 p.m. The first hiker arrived at Bridgewater 2.5 hours before the second hiker arrived at Amityville. When did the second hiker get to Amityville?

Figure 3. The rich problem.

Using the MATH process, we asked candidates to complete the tasks in Figure 4 below, related to the Bridgewater Problem either in pairs or individually.

Task 1: Begin by solving the Bridgewater problem with a partner, without referring to any outside materials. Show all work on separate paper. You are welcome to write work by hand.

Task 2: Examine student solutions. Identify student solutions that you would consider as (a) primarily numerical; (b) primarily geometric; (c) primarily algebraic without overt use of functions; and (d) primarily functional (overtly uses functions in solutions).

Task 3: Using one of the student samples as a starting point, construct a formal, answer-key quality solution to the Bridgewater Task that relies primarily on one of the following approaches: (a) numerical; (b) geometric; (c) algebraic (no functions used); or (d) functional approach (includes calculus-based approaches).

Task 4: Construct a modified version of the Bridgewater Task worksheet that fosters student exploration using either a numerical or geometric approach (no functions or significant equation solving required). Your worksheet should be no longer than 2 single-sided pages.

Task 5: Reflect on what you've learned while completing this activity. Identify one "big idea" you can take away from Mathematics Project 1 that will help inform your future teaching of mathematics at the high school level.

Figure 4. The five part assignment for candidates.

Results

In our analysis, we focus on Tasks 2–5 of the MATH process assignment. As previously mentioned, each step of the process encourages candidates to reconsider rich tasks from increasingly teacher-centric points of view. Task 1 puts the teacher candidates firmly in the position of doer of mathematics and produces examples of their own mathematics that the students can then compare to other people's mathematics. In Task 2, the candidates read through many examples of high school students' solutions to the task. The purpose of this task is for the teacher candidates to be confronted with many examples of how the problem can be solved including perspectives totally different from their own. In undertaking this task the students can begin the shift to thinking about the problem from another person's perspective. Task 3 requires candidates to internalise the idea of other perspectives and incorporate other viewpoints into their own work. The purpose of this task is to require student candidates to continue their shift from doer to teacher by thinking about how students work can be assessed. Task 4 provides candidates with an opportunity to reconfigure the problem to guide/scaffold exploration by the student. The purpose of this task is to require candidates to engage in task design in such a way that they have to think about how someone else would do the problem. Finally Task 5 requires the teacher candidates to reflect as in Loughran (2002) on the process they have undergone and to articulate their thoughts on the shift they are making from doer to teacher.

Analysing Student Work

Candidates were provided with a large range of solutions to the Bridgewater task produced by high school students at a local co-operating school. The work the candidates examined, therefore, was authentic work as produced by students engaged in the task. As we will see there was a certain power in this authenticity that affected the students in analysing the tasks. Rather than being told by an instructor that alternative approaches are important and to be encouraged, the candidates were able to see that there genuinely were many ways to approach the task and, furthermore, that the work of interpreting alternative solutions was far from straightforward.

Whereas teacher candidates tend to solve the Bridgewater task in fairly similar ways—typically using rates and variables to construct functions and equations that may be solved to determine an answer—high school students use a wide array of approaches. As will be shown below, the candidates' extensive experience with symbolic forms and function in their recent studies of calculus resulted in them privileging these approaches. Furthermore, their commitment to variable/functional approaches led to them having difficulties in interpreting the numerical and geometric approaches such as those depicted in the Figures 5, 6, 7, and 8 below. The teacher candidates' training as doers of mathematics led them to more abstract approaches than high school students' solutions, which were often quite performative in that they were based on creating stories where the walkers' actions at points in time are developed and then compared. High school students' approaches that were based on creating meaning and actions were not consonant with the teacher candidates' approaches that involved creating meaning based on mathematical abstraction.

In the solution shown in Figure 5, the path is given a fixed value. The student then chooses unit rates and associated distances for each hiker and scales the unit rate to three hours looking, experimentally for a total distance of the fixed length.

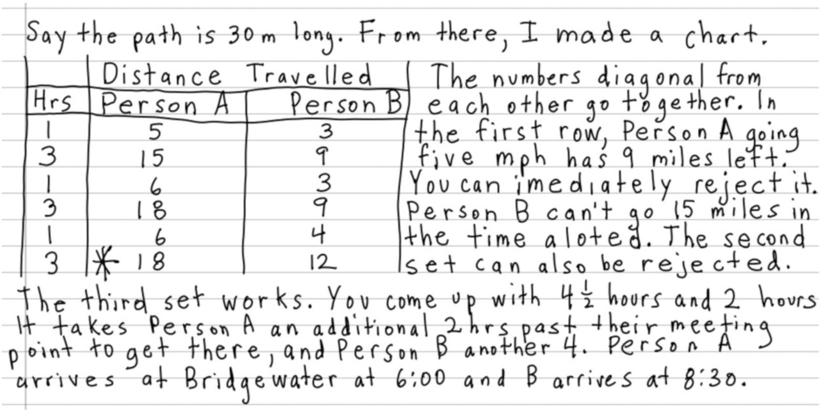

Figure 5. A numerical/tabular approach to the task.

The student solution shown in Figure 6 sets up two number lines 2½ units different in length and compares individual hiker's movement along the lines to find when they meet.

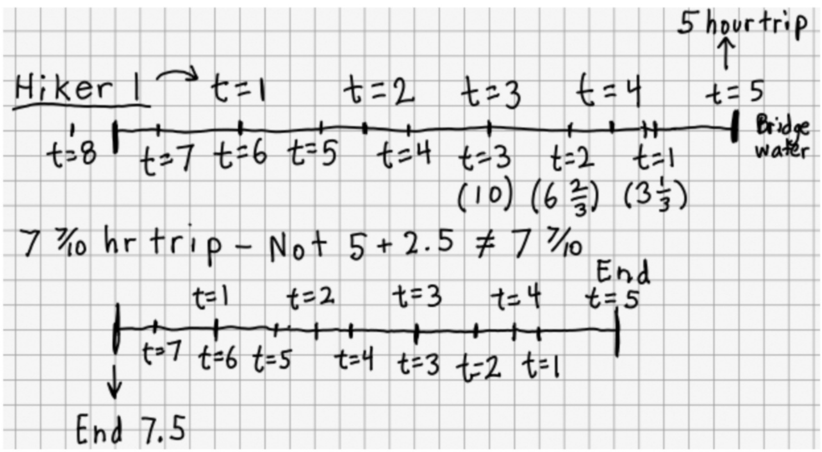

Figure 6. Student solution to Bridgewater task, highlighting a geometric number line approach.

In the solution shown in Figure 7, perhaps the most unorthodox, the student starts both hikers at the same time and draws line segments from left to right to represent the time of the first hiker's journey and then draws corresponding line segments from right to left for a journey which is 2½ hours longer, looking for when the lines will intersect at 4pm.

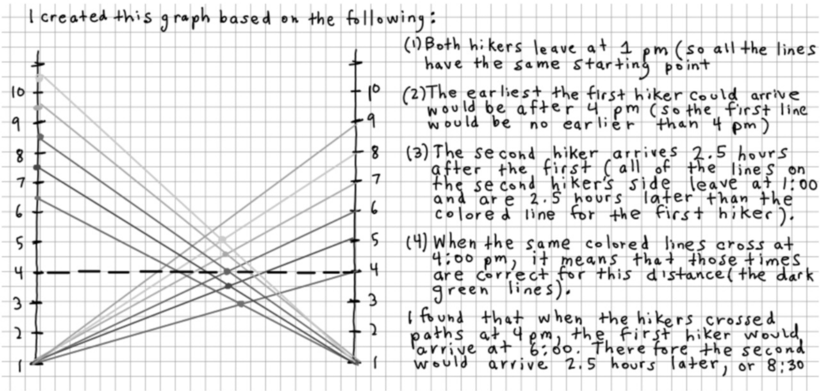

Figure 7. A solution with a graphical approach.

In the student solution shown in Figure 8, separate distance functions for the two hikers were added together to make a chosen fixed length were graphed (graph not shown) to find the meeting time.

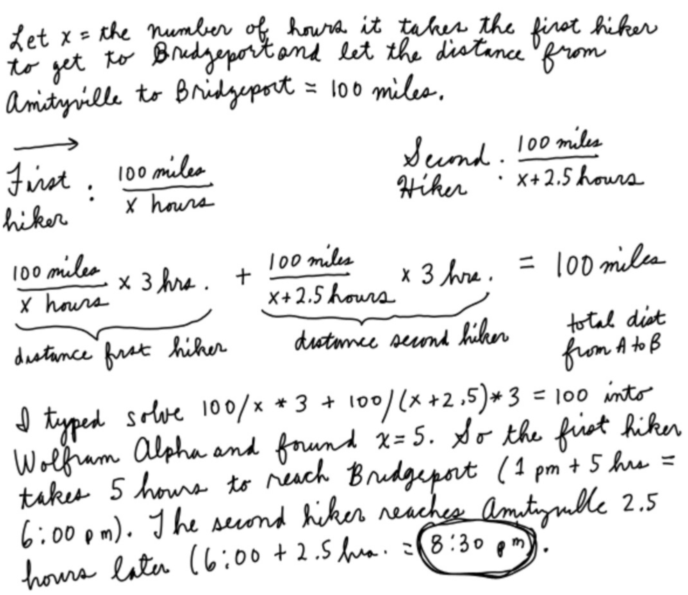

Figure 8. A solution using the intersection of graphs of functions.

Candidates' Reflections on Their Analysis of Student Work

> We had never had an experience like this where we were critically thinking about students' solutions, and this skill can be very helpful as we become teachers in the future. (Teacher Candidate, EDT 430)

Although many candidates were familiar with the notion that students solve problems in different ways, the visceral experience of seeing so many authentic high school student solutions, as opposed to manufactured solutions designed to make the point that many approaches are possible, resulted in many expressing surprise upon seeing the variety of solutions that students created for the Bridgewater problem. Nineteen of twenty reflections included comments expressing the sentiment that "after examining the student work it occurred to me that not only are there multiple ways to solve the problem, but there are different ways within the same basic approach". The candidates were being asked to "unpack" (Adler & Davis, 2006) processes of production; and "pace" (Silver & Thompson 2008), began to be "decentred".

As candidates take initial steps towards becoming a teacher, they begin to realise that nothing is clear-cut and that there are no "cookie cutters". Eight of twenty groups discussed difficulties that they experienced during the interpretations of student work. Until this point, candidates had only considered their own work and thought processes. They began to understand a new dimension of the transition from being a doer to being a teacher.

> Many of the solutions were hard to follow at first because of their seeming lack of logical procession ... we did not expect to see the vast array of methods that the students used, none of which approached the problems in the same way that we did.

It was apparent that students' reasoning was different from that of the candidates. The teacher candidates had learned that such problems could be solved using algebra and were inclined, in their own solving of the problem, to deploy algebraic methods without consideration of other methods. Eleven groups mentioned a similar issue. Some groups discussed their way of thinking or their solution and compared this to the students' ways of thinking or solutions. Other candidates mentioned the need to be more aware of different solutions, not just thinking about their own.

> We solved the problem algebraically and hadn't thought of another way doing so. The project showed me that as future teacher I need be aware of any and all possible solutions for a problem, because I'm bound to have at least one student who takes that approach.

We see in these responses that the teacher candidates are beginning to acknowledge the nature of the shift they must know undertake. The logic of their own approaches is based on mathematical abstraction and, having spent a great deal of their training using and appreciating the value of abstract approaches they find it challenging to interpret other approaches. Thus a logic that is based on "performing" the problem, as many of the high school students using numerical approaches do, appears to the teacher candidates as not following a logic.

Candidates also discussed more practical issues regarding the analysis of students' work. Five groups were surprised to see how much time it took to grade students' assignments and they could see the need for rubrics:

> In a sense the task introduced us to the reality of being a teacher, such that we are going to have to face the fact that the grading and the understanding of each individual student's work will take a significant of time.

In some ways it is perhaps surprising that the idea that someone else would approach the problem differently should be so powerful for the candidates. However, candidates are confronted by this fact in a very visceral way and in the context that all of their fellow candidates' solutions were very similar algebraic approaches.

Creating an Answer Key

As teacher candidates, the participants rely on their previous experiences as learners when developing solution keys for students to follow. Although candidates were told that "classroom ready" keys would be shared with students and their teacher, the candidates' work resembled exam or graded homework solutions more than materials produced for younger learners. Several common weaknesses in candidate solution keys included the lack of student guidance and scaffolding, complicated or incomplete use of mathematical notation, and ambiguous text accompanying mathematical notation.

We see here a clear distinction between the CK of the candidates in producing solutions and their unsophisticated MKT as they attempted to construct solutions that would be understandable by, and useful to, others. Several examples are provided in Figures 9–14 to demonstrate the common problems described below.

Lack of guidance/scaffolding. Candidates did not provide detailed explanations of how they built the mathematical model to solve the problem. Teacher candidates also failed to provide a detailed explanation regarding what the mathematical formulas or variables stand for in their models.

Incomplete mathematical notation. In the solution key in Figure 9, the candidate explains variable t, but the reader is left to guess what m, n and D represent.

Vague answer keys. Candidates often attempted to introduce graphical or diagrammatic aspects to solutions to help with visualising the problem, but not always with great success. Figure 10 represents an answer key with vague visual and with nothing to indicate what exactly the tick marks below the number line represent.

Problems with mathematical notation. Many candidates failed to pay adequately close attention to labeling the units. They also chose to use mathematical terms instead of contextual names. This is further evidence that the teacher candidates' approach is to abstract a problem to deploy "good"

Call the time that it takes George to get from Amityville to Bridgewater t. Therefore, Peter's time will be t +2.5. by our distance formula, we know the following:

D=tm

D=(t+2.5)n

Now solve these two equations for m and n:

$$m = \frac{D}{t}$$

$$m = \frac{D}{t+2.5}$$

Figure 9. Solution key with incomplete mathematical notation.

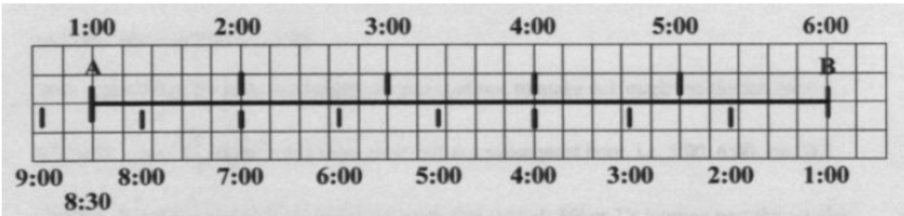

Figure 10. Answer key with an unclear visual image.

Now, we will use or table to find numerators and denominators that divide equal to 3 (the difference in hours travelled).

Numerator	Denominator	Divide	Answer
27	9	$\frac{27}{9}$	3
15	5	$\frac{15}{5}$	3
9	3	$\frac{9}{3}$	3
6	2	$\frac{6}{2}$	3

So now, using our equations above, we must find numerators that satisfy $\frac{d_1}{s_1} = 3, \frac{d_2}{s_2} = 3$.

Figure 11. Solution key with vague mathematical labels.

mathematics rather than use the context of the problem to perform the story and make meaning. Figure 11 shows the work of a candidate who chose to label the columns of their table with vague descriptors.

The use of non-standard variables and inconsistent use of variables were the other issues that might complicate the answer keys that candidates provided for their future students. Figure 12 shows an answer key with variables defined using superscripts. Here the use of superscripts is ill-advised since they look like exponents, which would be likely to confuse students.

$$\frac{D^1}{S^1} = 3hrs$$
$$\frac{D^2}{S^2} = 3hrs$$
$$\frac{D}{S^1} = \frac{D}{S^2} - 2.5hrs$$
$$D^1 + D^2 = D$$

D^1 = Distance of 1st hiker in 3 hours
D^2 = Distance of 2nd hiker in 3 hours
D = Total distance of trip
S^1 = 1st hiker constant speed
S^2 = 2nd hiker constant speed
T^1 = Time 1st hiker can go from A to B
T^2 = Time 2nd hiker can go from A to B

Figure 12. Solution key with complicated mathematical notation.

Ambiguous text. Here candidates knew what they were talking about, but they had difficulties in communicating that knowledge to a student. Some of their writing was not clear or was difficult to follow. Figure 13 presents an answer key with a very wordy explanation that might confuse students even more than it might help.

It stands to reason that combining the two distances stated above would result in total distance of the road they traveled due to their walking opposite directions on the same road. Further, we can say that the sum of each hikers' respective rates and times found above will also yield the total distance of the road each traveled due to those rates and times being equivalent to each respective distance described above. This results in the forming of the following equation:

(time of hiker$_1$) (rate of hiker$_1$) + (time of hiker$_2$) (rate of hiker$_2$) = total dist.

Figure 13. Answer key with wordy text.

In Figure 14, we see how the candidates start up the problem solving process by providing some guidance. But it might be confusing for students to understand what the teacher is asking them to guess due to lack of details and having very wordy text. More explicit guidance would be helpful here.

We know that Hiker #1 arrived at Bridgewater 2.5 hours before Hiker #2 arrived at Amityville, so we can conclude that Hiker #1 is walking at a faster rate than Hiker #2. This is also why the meeting point in the picture is closer to Bridgewater; Hiker #1 is traveling faster, therefore he will be closer to his destination at their meeting point than Hiker #2 will be to his destination.

At this point, you can proceed by guessing and checking. Use the formulas given at the top of the page. Before you start random guesses, try making educated guesses. Since we established that Hiker #1 is travelling at a faster rate than Hiker #2, you know to guess higher numbers for Hiker #1's rate compared to your guesses for Hiker #2's rate.

Figure 14. Answer key with wordy and ambiguous text.

Reflection on the Creation of a Solution Key

In discussing their experiences in the development of the solution key, candidates identified the main struggle as deciding how to cater to the diverse body of students without providing multiple keys. As one candidate noted, "we also learned that it would be difficult to create an answer key based off of [sic] one method. It is impossible to use all the methods to make an answer key, but by choosing only one method it feels as though we are leaving out so much great work that the students could learn from." Clearly, the creation of a solution key provided another experience for the candidates in thinking not just about how they would do a problem but how others would do the problem and how they would evaluate and assess that work. This process allowed us another opportunity for the candidates to see gaps in their own MKT particularly in the assumptions they make about a learner's ability to understand particular representations and understand particular notations.

Modification of the Worksheet

When reconfiguring the worksheet candidates tended to provide more explicit structure for their students, as required by the task. The main guidance of 18 worksheets out of 20 included was the table for students to use for their educated guesses. Figure 15 displays an example of a table that included a structure whereby students make an initial guess and then respond to questions guiding students toward a next, more informed, guess. However, some of the teacher candidates chose not to provide a sample guess in the table for their students.

R_1	R_2	TD (using the equation from instruction 2)	Require difference in total travel time by the problem (in hrs.)	Resulting difference in total travel time from guesses (in hrs.) (using equation in instruction 3)	Are numbers in the same row of columns four and five equal?
3	4	3(3) + 3(4) = 21	2.5	(21/4) – (21/3) = -1.75	No
3			2.5		
3			2.5		
3			2.5		
3			2.5		
3			2.5		

Figure 15. Tables for educated guesses.

We believe that when preparing a worksheet for students, candidates were more pedagogically-oriented than was the case with their solution key preparation. For instance, candidates asked students to explain their answers in their revised materials. One group asked students to use technology to solve the problem. Another group included a rubric for their students to show how different parts of the problem were going to be evaluated differently. Two groups focused on helping students to extend the solution for different rates. Nevertheless, there were still some problems with some of the worksheets that candidates created. For instance, many worksheets included vague questions, variables, or tables.

A particular instance was a worksheet asking students to create respective formulas for the hikers but was not clear which distance was to be used: "Now create two formulas using distance and time to find George and Peter's rates". We see here another gap in the candidates' MKT in their unsophisticated and underdeveloped attempts to scaffold students' learning. It is also interesting to note that despite the plethora of approaches that the candidates saw in the student work, the majority of worksheets were oriented around guiding the students through the development of an algebraic approach. We can argue that the candidates moved from being concerned about their own approach towards inviting others to take their approach, but have not moved all the way to constructing tasks in a way that does not privilege their own approach to the problem. The evidence suggests that the candidates still think of an algebraic approach as the best and "most mathematical."

Other problematic areas in the creation of the worksheets were formatting issues. A sample is shown in Figure 16. Some of the candidates used unusual fonts that made it more difficult to read. Others did not leave any space for students put their answers. Recurrent problems were: not using units; the choice of seemingly arbitrary distances; and the use of non- standard variables.

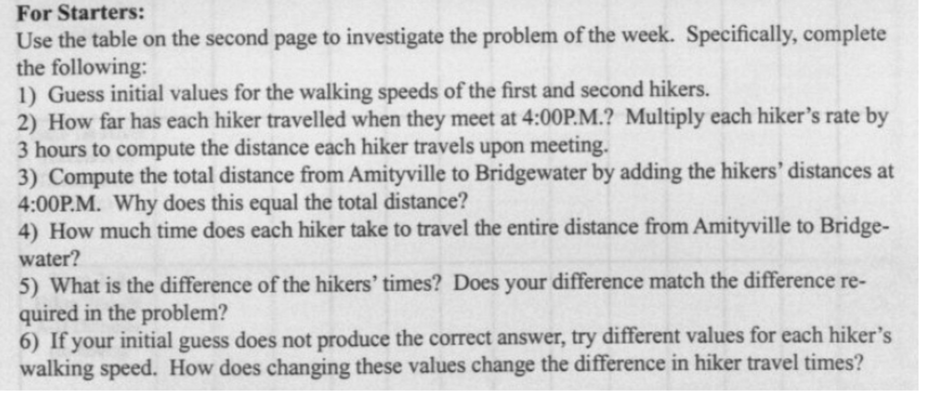
For Starters:
Use the table on the second page to investigate the problem of the week. Specifically, complete the following:
1) Guess initial values for the walking speeds of the first and second hikers.
2) How far has each hiker travelled when they meet at 4:00P.M.? Multiply each hiker's rate by 3 hours to compute the distance each hiker travels upon meeting.
3) Compute the total distance from Amityville to Bridgewater by adding the hikers' distances at 4:00P.M. Why does this equal the total distance?
4) How much time does each hiker take to travel the entire distance from Amityville to Bridgewater?
5) What is the difference of the hikers' times? Does your difference match the difference required in the problem?
6) If your initial guess does not produce the correct answer, try different values for each hiker's walking speed. How does changing these values change the difference in hiker travel times?

Figure 16. Formatting issues.

Candidates' reflection on modifying the task. In reflecting on their worksheet design some candidates discussed their struggle to balance providing structure in the worksheet to guide students while not providing so much structure that the students are simply led to the answer without having to make any decisions. They also struggled to serve diverse learning styles.

The creation of the new handout we made forced us to consider the student perspective. This allowed us to read what we created as a student might, which allowed us to troubleshoot our worksheet by making it more clear what we wanted students to do through the use of unambiguous language and very clear directions.

The quote below provides evidence of the change that some candidate teachers underwent through the creation of a worksheet with students firmly in mind.

> Revising the worksheet was a challenge for me … I tried very hard to make sure that students are given hints, examples, and even answers if they look hard enough.

This task provides a particularly interesting challenge to the candidates in terms of thinking about scaffolding and how they create trajectories in learning tasks that can facilitate student learning.

A possible future direction in implementing the MATH process would be to ask candidates to create different worksheets which ask the students to work on the task in different ways: algebraically, numerically and diagrammatically.

Discussion

Returning to Brown and Borko (1992), we recall they argued that there are three important factors in the process of "learning to teach": (a) the influence of the content knowledge on a teacher candidate's conception of mathematics; (b) the teacher candidate's efforts to learn pedagogical content knowledge; and (c) the difficulties for teacher candidates in acquiring pedagogical reasoning skills. The MATH approach in which the teacher candidates in this study were engaged promoted specific development in each of these areas. The use of authentic student solutions to a task the teacher candidates had worked on themselves and the fact that those student solutions presented so many alternative perspective (most of them not even considered by the majority of the teacher candidates) served as a surprisingly visceral demonstration of the need they now had, as teacher candidates, to think about mathematical work in an entirely new way. Furthermore the dissonance created, in particular by the examination of the authentic student work, facilitates the "unpacking" (Adler & Davis, 2006) and "decentering" (Silverman & Thompson, 2008) that we see as vital steps in the teacher candidates' shift from "doer" to "teacher."

Candidates' Content Knowledge is engaged not only in solving the problem themselves but also through the analysis of numerical, geometric and functional approaches to solving the problem in addition to the algebraic approach on which most of them relied.

Their Pedagogical Content Knowledge is engaged both in their attempts to create a solution key and their reconfiguring of the problem as an exploration task. For both of these tasks the candidates had to engage with the idea of how someone else is thinking about the problem, how to anticipate other people's responses and how to design a task which can account for many approaches not just their preferred one.

Finally, pedagogical reasoning skills are engaged throughout, particularly in Task 2, the analysis of student work, where the candidates have to reason through and understand a student's thinking and think about how they could discuss the solution with the student.

Even though we are becoming teachers, all of our classes so far … have never let us experience the role as a teacher … This was the first real chance we had to look at student work.

These candidate teachers were beginning to see themselves getting up from students' chairs and going to the front of the classroom.

The key juncture for a candidate in the transition from being a "doer of mathematics" to being a "teacher of mathematics" is learning how to engage with, and to understand, someone else's mathematics. A key skill that candidates must develop is the capability to understand and interpret multiple solution paths (algebraic, numerical, diagrammatic, or geometric) for mathematics tasks. The evidence of our study is that extended engagement with authentic alternate solutions to rich tasks with which the candidates themselves have already engaged is a powerful tool in scaffolding each candidate's path in the vital transition from solving tasks on their own terms for themselves (being a doer) to understanding and engaging with other's attempts to solve tasks (being a teacher). The MATH approach also activates and develops teacher candidates' KCS (Hill, Ball, and Schilling, 2008).

We believe that all five steps in the MATH taken together provide a powerful mechanism for providing candidates with a dissonant experience which forces them to reflect on their own understanding of mathematics but, more importantly, provides a framework to move candidates from doers or learners of mathematics to being teachers of mathematics. As we mentioned in our conceptual framework, this process requires a life-long learning attitude: "Not only can students learn from teachers, but we both learned a lot from these students."

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