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Pre-service Secondary Mathematics Teachers Making Sense of Definitions of Functions

Joshua Chesler
California State University, Long Beach

Definitions play an essential role in mathematics. As such, mathematics teachers and students need to flexibly and productively interact with mathematical definitions in the classroom. However, there has been little research about mathematics teachers' understanding of definitions. At an even more basic level, there is little clarity about what teachers must know about mathematical definitions in order to support the development of mathematically proficient students. This paper reports on a qualitative study of pre-service secondary mathematics teachers choosing, using, evaluating, and interpreting definitions. In an undergraduate capstone course for mathematics majors, these future teachers were assigned three tasks which required them to (1) choose and apply definitions of functions, (2) evaluate the equivalence of definitions of functions, and (3) interpret and critique a secondary school textbook's definition of a specific type of function. Their performances indicated that many of these pre-service mathematics teachers had deficiencies reasoning with and about mathematical definitions. The implications of these deficiencies are discussed and suggestions for teacher educators are proposed.

Introduction

Definitions matter in mathematics. They introduce ideas, they describe objects and concepts, they identify fundamental and essential properties of mathematical objects, they support problem solving and proof, and they facilitate communication of mathematics (Zaslavsky & Shir, 2005). Accordingly, the United States' *Common Core State Standards for Mathematics* [CCSSM] acknowledges the importance of definitions in the mathematics education of all K-12 students (Common Core State Standards Initiative, 2010). Within their eight "Standards for Mathematical Proficiency", the CCSSM note that mathematically proficient students understand and use definitions in constructing arguments, in their reasoning, and in communication about mathematics. However, there has been relatively little research either on student learning or on teacher knowledge of the roles and uses of mathematical definitions (deVilliers, 1998; Moore-Russo, 2008; Vinner, 1991; Zaslavsky & Shir, 2005). Moreover, the research that does exist on in- and pre-service mathematics teachers interacting with mathematical definitions indicates that many have deficiencies in this area (e.g., Leikin & Winicki-Landman, 2001; Linchevsky, Vinner, & Karsenty, 1992; Moore-Russo, 2008; Vinner & Dreyfus, 1989; Zazkis & Liekin, 2008).

The study reported herein is an examination of pre-service secondary mathematics teachers (PSMTs) choosing, analysing, evaluating, and using definitions in a mathematics capstone course taken in the final semester of their undergraduate mathematics program. The broad goal is to help illuminate the task and challenges of training PSMTs to prepare mathematically proficient students who reason from, with, and about definitions, as envisioned in the CCSSM and elsewhere. The present focus is on pre-service secondary teachers, however, it should be noted that elementary teacher preparation with mathematical definitions is likewise important. For example, the mod4 project at University of Michigan recently published professional development materials entitled *Using Definitions in Learning and Teaching Mathematics* for elementary teachers (mod4, 2009). The activities focus on identifying the roles and emphasizing the importance of definitions in teaching elementary school and, more generally, in mathematical reasoning and in the discipline of mathematics. Furthermore, they address the question, "What makes a good mathematical definition?" (p. 1). This question is one which the PSMT in the present study had to confront and, fortunately, it is one about which there is some clarity.

A mathematical definition must not be self-contradicting or ambiguous; it must be invariant under choice of representation and it must be hierarchical (i.e., based on prior concepts) (Zaslavsky & Shir, 2005). Furthermore, mathematical definitions are arbitrary; for any particular object or concept, there are many equivalent ways to define it. On this point, Winicki-Landman and Leikin (2000) note that, "teachers' professional development should include activities focusing on the issue of equivalent and non-equivalent definitions" (p. 21). Zaslavsky and Shir (2005) characterize definitions as either (1) procedural, describing how an object is constructed, or (2) structural, identifying essential properties of an object. In addition to these mathematical considerations, there are didactic considerations when evaluating a definition for use in a classroom setting. This interplay between mathematical and classroom considerations is at the heart of the present study which investigates 23 PSMTs working with definitions about functions and identifies some of the challenges they encountered.

Literature Review

Teachers must draw upon various types of knowledge to effectively interact with mathematical definitions in the classroom. The Mathematical Knowledge for Teaching (MKT) model proposed by Hill, Ball, and Schilling (2008) provides a useful framework for unpacking this sort of teacher knowledge and distinguishes between subject matter knowledge and pedagogical content knowledge (PCK). For instance, teachers must have sufficient PCK to choose age-appropriate definitions, or to respond to student questions about or work with definitions. The present focus, however, is more on PSMTs' subject matter knowledge; particularly, there is a specialized content knowledge (SCK) which teachers must draw upon to interpret, evaluate, choose, or use definitions. This includes, but is not limited to, an awareness of the features of mathematical definitions discussed above. This type of knowledge overlaps with what Ernest (1999) described as the meta-mathematical knowledge about definitions; i.e, the largely tacit norms and standards of definition use within the mathematical community.

Educational research and perspectives on mathematical definitions complement this outlook on teacher knowledge. Notably, Tall and Vinner (1981) drew a contrast between the concept image as the "total cognitive structure" (p.

152) that an individual associates with a concept and the concept definition as the words used to describe a concept. They further distinguished between a personal concept definition constructed by the individual and formal concept definitions accepted by the mathematics community. They noted that concept images need not be coherent and that concept images can be, and often are, in conflict with concept definitions. Indeed, Vinner and Dreyfus (1989) found this to be the case in their survey of college students' and junior high teachers' conceptions of mathematical functions. The 307 respondents were asked to define functions; of them, 82 supplied a Dirichlet-Bourbaki definition of functions as a correspondence between two nonempty sets that assigns exactly one element of the second set (the co-domain) to every element of the first set (the domain). However, when working on other function tasks, these 82 respondents displayed inconsistent behaviour; 56% of them did not use this conception of functions when answering other questions about functions. This was described as the respondents having potentially conflicting cognitive schemes for which concept images and definitions were not mutually supportive; a phenomenon also described as compartmentalization (Vinner, Hershkowitz, & Bruckheimer, 1981). Vinner (1991) advised that, in negotiating these conflicts, the roles of definitions in a mathematics class should be determined by the educational goals.

Other researchers have demonstrated this difficulty interacting with definitions amongst pre- or in-service mathematics teachers. Linchevsky et al. (1992) reported that out of a group of 82 pre-service teachers, all of whom expressed an interest in potentially teaching junior high school mathematics, only 21 were "aware of the arbitrariness aspect of definition" (p. 53). Moore-Russo (2008) found that, among the 14 pre- and in-service secondary mathematics teachers in her study none had any prior experience with definition construction. She reported that definition construction activities helped the study participants develop a deeper understanding of slope. Leikin and Winicki-Landman (2001) reported on professional development activities for secondary mathematics teachers which focused on "what is definition" and "how to define" in order to deepen the participants' subject matter and meta-mathematical knowledge (p. 63). The researchers noted that many teachers were unaware of the arbitrariness aspect and of the consequences of particular definition choices. Elsewhere, they described the teachers' strategies for evaluating the equivalence of definitions as either based on logical relationships between the definitions ("the properties strategy") or by comparing the sets of objects determined by each definition ("the sets strategy"). A third, but rarely-used, strategy was based on referencing a representation of the object ("the representation strategy") (Leikin & Winicki-Landman, 2000, p. 25). Shir and Zaslavsky (2001) noted inconsistencies amongst mathematics teachers evaluating the equivalence of definitions of squares; the teachers were particularly unsuccessful in evaluating procedural definitions. The 24 teachers in their study considered both mathematical and pedagogical concerns in determining equivalence. Zazkis and Leikin (2008) reported on pre-service secondary mathematics teachers creating definitions of squares; of the 140 definitions provided, about 40% were found to be inappropriate. Other studies have documented limited understanding of definitions among pre-service elementary teachers interacting with geometrical objects (e.g., Chesler & McGraw, 2007; Fujita & Jones, 2007; Pickreign, 2007).

The studies referenced in the preceding paragraph indicate that many pre- and in-service teachers have deficiencies in their understandings of definitions. However, in general, there has been relatively little attention given to definitions in mathematics education research (deVilliers, 1998; Moore-Russo, 2008; Vinner, 1991; Zaslavsky & Shir, 2005). Notable amongst the few studies which examine student understanding of definitions at the K-12 level is that of Zaslavsky and Shir (2005) who studied conceptions of mathematical definitions among four advanced 12th-grade mathematics students. They noted that students evaluated definitions according to mathematical, communicative, or figurative considerations. That is, in determining if a proposed definition is acceptable, they focused, respectively, on logical concerns, on clarity, or, in the case of geometrical definitions, on some prototypical mental picture(s) of the object being defined. Certainly, there is some overlap between these three types of considerations and the strategies described by Leikin and Winiki-Landman (2000); the "properties strategy" perhaps aligns with mathematical considerations and the "representations strategy" with figurative considerations. Zaslavsky and Shir also reported that, in evaluating definitions, the students justified their responses either by referencing examples or by referencing features or roles of the definitions.

Despite the few studies that directly address conceptions and understanding of mathematical definitions, many have acknowledged K-12 students' meaningful interactions with definitions as important or essential. De Villiers (1998) wrote of definition construction as "a mathematical activity of no less importance than other processes such as solving problems, making conjectures, generalizing, specializing, proving, etc., and it is therefore strange that it has been neglected in most mathematics teaching" (p. 249). Ouvrier-Buffet (2006) and Harel, Selden and Selden (2006) likewise noted that constructing definitions can foster both students' and teachers' productive reflection on mathematics and can deepen teachers' insight into student understanding. Zaslavsky and Shir (2005) similarly noted that considering alternate definitions can help refine students' conceptual understanding.

Thus, the limited research on in- and pre-service mathematics teachers' indicate that many struggle with constructing definitions, evaluating alternative definitions, and using definitions to reason and justify. However, the importance and value of definitions throughout mathematics education has been widely acknowledged both by researchers (e.g., Harel et al., 2006; Vinner, 1991; Winicki-Landman & Leikin, 2000; Zaslavsky & Shir, 2005) and in standards documents (Common Core State Standards Initiative, 2010). Interactions with definitions are built into the secondary mathematics teacher's role as she/he must evaluate, interpret, and model the use of definitions. Further complicating this, the definitions which teachers encounter in curricular materials often do not foster conceptual understanding or help build a logical foundation for future mathematics studies

(Harel & Wilson, 2011). Vinner (1991) advises teachers and textbook writers to be cognizant of the "cognitive power that [a] definition has on the student's mathematical thinking"; something, which he warns, is often neglected (p. 80).

Methodology

Data are comprised of student work on three problems assigned in a capstone course in the mathematics department at a large masters-granting university in the western United States of America. The author of this paper was also the instructor. The course is required for all undergraduate mathematics majors who intend to be secondary mathematics teachers. Of the 23 students enrolled in the course, 19 were in their last semester of undergraduate study (working toward a BSc in Mathematics with an option in secondary education) and 4 had already completed an undergraduate mathematics degree (3 were enrolled in the teaching credential program, 1 was enrolled as a graduate mathematics student). Throughout this paper, the students are referred to as pre-service secondary mathematics teachers (PSMTs) as each intended to (or at least was keeping the option open to) follow that career path. Each of the three problems presented to the students required them to answer questions about definitions of functions. Two of the questions were assigned as homework problems and one was assigned on a take-home exam. Not all students answered each of the three problems. Students were encouraged to work together on homework problems (though there is no data about the extent to which this occurred) and were forbidden to collaborate on the take-home examination.

The three problems were not specifically designed as research instruments. However, in the prior semester of this course, similar problems had yielded interesting results that inspired data collection and a refinement of the problems in an attempt to explore the themes described above. The analysis process was an iterative search for patterns through coding of student responses (Coffey & Atkinson, 1996). The first round of coding recorded the degree to which students successfully completed the required tasks which explored PSMTs' knowledge of the roles of definitions, the arbitrariness of definitions, and the pedagogic dimension of definition choice. Subsequent rounds were driven by emergent themes. Further details about coding strategies are embedded in the Results section of this paper.

Results

This section explores and is organized around the PSMTs' responses to the three problems that required them to choose, use, evaluate, and analyse definitions.

Problem 1

On a homework assignment from the second week of the semester, students were required to select a definition for function and then to use that definition to justify why sequences are functions (see Figure 1). At the time of this assignment, some class time had been devoted to discussions of functions and their representations; however, sequences had not been discussed nor had there been explicit discussion about the roles of mathematical definitions.

> This question has 2 parts:
>
> a) Write down a definition of functions. You may use a definition you're familiar with or you may find one somewhere but, in either case, note the source of your definition.
> b) Use this definition to justify why sequences are functions.

Figure 1. Problem 1

The definitions that students chose came from various sources such as websites, textbooks, and dictionaries. Nineteen of the 21 students for whom data were collected described functions as mappings, rules, relations, correspondences, or relationships between sets, variables, or inputs/outputs. In general, students chose correct definitions, which aligned with the Dirichlet-Bourbaki definition, yet their choices, often not in their own words, gave little insight into their understandings of functions or of mathematical definitions. However, the choices that students made are not of primary focus herein; see Vinner and Dreyfus (1989) for a categorization of students' definitions of functions. Though, it is noteworthy that two of the 21 students' chosen definitions defined functions as types of equations; these definitions were separated from the others because, even at the high school level, not all functions can be defined by equations. For example, PSMT #5 adapted a definition which was labelled as a "working definition" on a website titled "The Definition of a Function" (Dawkins, 2011). Both students described functions as equations in which an x is "plugged in", and a unique y is the result. Neither of these two students answered part (b) correctly and were unable to identify the domain or what equation would determine the terms of the sequence (which both identified as "the y's").

Indeed, more insight was gained from part (b) as students attempted to use their chosen definitions. In order to determine how or if students used their definitions, focus was given to the type of object which each student defined a function to be. Each definition was in the form, or could simply be reformulated as the form, "A function is a [direct object]". If the student either explicitly or implicitly referenced that object in part (b) then their response was coded as having referenced the object of the definition. For example, PSMT #13, who defined a function as an "association", noted that each "element" of a sequence has "a specific number associated with it" and gave a clarifying example. Since this student used a verb-form of the appropriate object, his response was coded as having referenced the object. An implicit use of the object occurred if, for example, functions were defined as relations in part (a) and reference was made to ordered pairs in part (b); two students did this.

Responses were also coded as either correct or incorrect; a correct answer would need to, in some way, identify the domain and range. Table 1 summarizes the results. Only five of 21 students referenced the object of their definition in part (b) and got the answer correct; the majority of correct answers did not make implicit or explicit reference to the definition supplied in part (a). Six students referenced their definition yet gave an incorrect answer. PSMT #1, for example, described functions as types of relations and, though he referenced that definition, his answer for part (b) was incorrect: "Since in (a) we get a set of ordered pairs, a sequence $\{a_n\}$ is also a set of numbers written in a definite order." Another student, PSMT #12, who described functions as relations, also made reference to ordered pairs yet incorrectly and vaguely wrote, "Sequences are functions because each term, which lives in the domain, is paired with exactly one element in the range".

Table 1
Summary of Responses to Problem 1

	Referenced Object	No Reference to Object	Total
Correct	5	7	12
Incorrect	6	3	9
Total	11	10	

Seven students had correct answers for part (b) but did not reference the object in their definitions or, as was the case with PSMT #9, chose a definition that was too restrictive and was at odds with an otherwise correct answer in part (b). She supplied a definition, attributed to Dirichlet and historically significant, which, in the context of sequences, is too restrictive, as it required the domain to be defined on an open interval:

> y is a function of a variable x, defined on an interval $a < x < b$, if to every value of the variable x in this interval there corresponds a definite value of the variable y. Also, it is irrelevant in what way this correspondence is established. (Luzin (1998) provides an extended discussion of this definition.)

If a PSMT response was coded as "No Reference to Object", it did not mean that there was no reference to the definition at all. PSMT #19, for example, described functions as "rules" which "link" elements of sets and, though she made no explicit or implicit references to rules in part (b), she correctly identified what was being linked: "any element of [the set of Natural Numbers] can be linked to one and only one element of the sequence". In general, in the analysis of student responses, it was difficult to separate their knowledge of functions from their habits using definitions. This challenge will be revisited in the Discussion section.

Problem 2

During class, students had, in groups, compared and discussed several definitions of functions. Though not formalized, there was a discussion about what constitutes equivalent definitions. This was intended to be an activity that would both deepen their understanding of functions and help future teachers develop a more critical and analytical view of textbook definitions. A follow-up activity, Problem 2 (see Figure 2), was assigned as homework in the third week of the semester in which two definitions were to be compared. The primary intention was for students to notice that Definition I requires the domain and range to be sets of numbers whereas Definition II is less restrictive.

Here are some definitions of functions:

i) "A function is a rule that takes certain numbers as inputs and assigns to each a definite output number." From *Calculus* by Hughes-Hallett et al. (2006)

ii) "A function is a special type of relation in which each element of the domain is paired with exactly one element of the range." Relation had been previously defined as "a set of ordered pairs ... The domain of a relation is the set of all first coordinates from the ordered pairs, and the range is the set of all second coordinates of the ordered pairs." From *Algebra 2* by Holliday et al. (2005)

Answer these questions:

a) Are these definitions equivalent? Explain.

b) What is the word "special" referring to in the second definition? (Hint: Think about what is meant when we say that a square is a special type of rectangle.)

Figure 2. Problem 2

Of the 22 students who answered this question, 12 said that the definitions were equivalent, one said "yes and no", and nine said that they were not equivalent. Of these nine, only five attributed the lack of equivalence to the sets of numbers required in Definition I. The other four students said the definitions were not equivalent because, as PSMT #3 put it, Definition I "fails to clearly state that each input is assigned to a unique output". That is, the "definite output" in Definition I was not interpreted as a requirement for a "unique" output. PSMT #20 said that the definitions are "equivalent in one sense and not equivalent in another sense" and was perhaps distracted by the fact that Definition I came from a calculus textbook; he stated that this definition allowed for multivariate functions whereas Definition II did not. The success rate for part (b) was better, though five out of 22 students provided incorrect answers. These five students all erroneously made a connection between the word "special" and the condition that each input is paired with a unique output.

Problem 3

Problem 3 (see Figure 3) was assigned on a take-home examination distributed in the fourth week of the course. The intention was for students to engage in an authentic activity for secondary mathematics teachers, analysing a definition from a mathematics textbook. The definition was chosen because it was from an actual high school textbook and because it has some interesting issues; namely, the definition is dependent upon choice of representation and the condition "$q(x) \neq 0$" may not be presented with sufficient clarity.

Here's a quote from the *Glencoe Mathematics Algebra II* textbook (Holliday et al., 2005, p. 485):

A **rational function** is an equation of the form $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomial functions and "$q(x) \neq 0$". (p. 485)

On page 60 of (Cooney, Brown, Dossey, Schrage, & Wittmann, 1996), Mr Washington gives the following (in the chart) as an example of a rational function: $3x^{-3} + 2x^{-1} - 5x^2$.

The equation $y = 3x^{-3} + 2x^{-1} - 5x^2$ is not of the form $f(x) = \frac{p(x)}{q(x)}$.

a) Using the Glencoe definition and an equivalent statement Mr Washington's function, show that $y = 3x^{-3} + 2x^{-1} - 5x^2$ is a rational function.

b) Answer just one of these two related questions:

   i) Change the Glencoe definition so that it is more clear that functions such as $y = 3x^{-3} + 2x^{-1} - 5x^2$ or $y = \frac{5x+2}{x^2-3} + x^3$ are rational functions.
   (The more minor the change the better.)

   ii) What does a student need to understand to be able to realize that $y = 3x^{-3} + 2x^{-1} - 5x^2$ is a rational function, even though it is not written as the ratio of two polynomials?

a) Consider the Glencoe definition and do both of the following related questions/tasks:

   i) Explain why this definition includes the condition that "$q(x) \neq 0$".

   ii) What is meant by "$q(x) \neq 0$" in the definition? Keep in mind, as you construct your answer, that $f(x) = \frac{1}{x}$ is a rational function (because the numerator and denominator can both be thought of as polynomial functions) but the denominator of $\frac{1}{x}$ is zero sometimes! Also keep in mind that the author(s) of that definition thought that the condition "$q(x) \neq 0$" was necessary – so your answer should probably not imply that the condition was unnecessary.

Figure 3. Problem 3

All 22 students correctly answered part (a), which was intended to scaffold the other parts. Eight students chose to answer b) part i), six of these students changed the definition to something like the following:

> A **rational function** is an equation which can be written in the form $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ and are polynomial functions and $q(x) \neq 0$.

The other two students wrote something that was inaccurate; PSMT #20 wrote something incorrect ("Every polynomial function is a rational function written in the form …"), and PSMT #8 wrote something which was also dependent on representation. Of the 14 students who answered b) part ii) nine responded in a way similar to PSMT #3, writing that a student would need to "realize that $3x^{-3} + 2x^{-1} - 5x^2$ can be rewritten as the quotient of polynomial expressions". Four listed the skills that a student would need to manipulate the expression and one student noted that the field of rational expressions is closed.

The verbosity of part c) was an attempt to give the PSMTs enough clues to reason through why "$q(x) \neq 0$" is in the definition. Only six out of 22 students noted that this condition meant that $q(x)$ could not be the zero polynomial. This may provide more insight about the communicative power of the textbook's definition than about the PSMTs' knowledge. For example, a definition of rational functions is communicated with greater clarity in a college-level algebra textbook:

> A rational function is a function that can be put in the form $f(x) = \frac{a(x)}{b(x)}$, where $a(x)$ and $b(x)$ are polynomials, and $b(x)$ is not the zero polynomial (McCallum et al., 2010, p. 407)

Furthermore, this definition has the advantage of paralleling a common definition of rational numbers (i.e., A rational number is a number that can be put in the form $\frac{a}{b}$, where a and b are integers, and b is not zero. The other 14 students answered that the "$q(x) \neq 0$" condition in the Glencoe definition was included as a domain restriction.

Discussion

It was often difficult to determine the causes of student errors on the three problems. Were their incorrect answers the result of deficiencies in subject matter knowledge about functions, in general, more meta-level, knowledge about mathematical definitions, or in some combination of both? There are, however, some instances of relatively greater clarity. For example, 12 PSMTs correctly answered part b) of Problem 1 (Why is a sequence a function?) but seven of them did not reference the object which they defined function as even though they were explicitly asked to use their definition. Though this was perhaps an imperfect way of determining if a PSMT had "used" the definition, the results align with what Vinner and Dreyfus (1989) reported; they found that more than half of the students who gave a Dirichlet-Bourbaki definition of function did not use that definition when answering other questions about functions. Their description of this as a gap between concept image and concept definition is certainly relevant to the PSMTs who made these errors.

However, it is also likely that many of these PSMTs lacked the appropriate meta-mathematical knowledge about the roles of definitions in mathematics. Student work on Problem 1, in particular, may indicate that the relationship between this meta-mathematical knowledge and subject matter knowledge may be both complicated and context-dependent. For example, unlike the students who correctly justified why a sequence is a function, the majority of students (6 out of 9) who incorrectly answered that part of Problem 1 actually did reference the object of their definition. That is, they knew and conformed to that convention of "using a definition" but fell short on their content knowledge. Another notable example may be PSMT #5's choice of a definition which was clearly marked as "a working definition"; she may not have understood the limitations of such a definition. It would be worthwhile to further examine and explicate the relationship between subject matter knowledge (e.g., functions) and meta-mathematical knowledge (e.g., the role of definitions).

A similar question can be formulated about the relationship between pedagogical content knowledge (PCK) as conceptualized in (Hill, Ball, & Schilling, 2008) and knowledge of the role of definitions. By the nature of their craft, mathematics teachers interpret, model the use of, and build upon definitions in their instruction. They also may need to reconcile equivalent (or, at times, non-equivalent) definitions of the same object that appear in different curricular materials or in student work. The three-capstone problems explored PSMTs using, choosing, comparing, and evaluating definitions; for mathematics teachers, these are didactic actions, which could help or hinder student learning. For example, on Problem 3, only six out of 22 PSMTs correctly interpreted the "$q(x) \neq 0$" condition in a definition of rational functions from a high school textbook. On part b) of Problem 2, despite the hint, a quarter of PSMTs did not know what the word "special" meant in the context of "a function is a special type of relationship"; this is similar to what Chesler and McGraw (2007) noted about pre-service elementary teachers' difficulty interpreting the phrase "a special kind of". Furthermore, some PSMTs in this study had difficulty choosing a definition of function that could support required tasks (as with PSMT #9 on part b) of Problem 2).

Indeed, choice of definition mattered. Of the five students who described functions as types of "rules" in Problem 1, only one of them made any attempt to describe how a sequence can be thought of as a type of "rule". Perhaps defining functions as a different, more clearly defined type of object would have better supported the follow-up task. Of the 11 PSMTs who referenced the object in their definition in part b) of Problem 1, five of them used a verb-form of the object to explain why sequences are functions (e.g., if a function is an association then a sequence "associates"). It is possible, and worthy of study, that definitions which accommodate this action-object connection help narrow the gap between concept image and concept definition or even between action and object levels of understanding (Dubinsky & McDonald, 2001).

Unfortunately, the definitions that appear in textbooks do not always support harmony between concept image and definition. Harel and Wilson (2011), in reviewing a high school textbook, lamented that, "it is difficult to learn from this text what a mathematical definition is or to distinguish between a necessary condition and a sufficient condition. Students are also expected to discover definitions given pictures as hints" (p. 826). This was offered as one of many examples of "the sorry state of high school textbooks". Indeed, there is a difference in clarity between the two textbook definitions for rational functions which were presented above; the definition from the high school textbook (Holliday et al., 2005) had issues with two essential properties of a good definition: invariance under choice of representation and non-ambiguity. Choice of definition can support or undermine both teaching and learning.

Indeed, the PSMTs' performances on the three tasks highlight the notion that the definitions which these future teachers encounter in their classrooms, and how the PSMTs interpret and use these definitions, will be impactful. As Vinner (1991) wrote,

> Definition creates a serious problem in mathematics learning. It represents, perhaps, more than anything else the conflict between the structure of mathematics, as conceived by professional mathematicians, and the cognitive processes of concept acquisition. (p. 65)

Moreover, as exemplified by the textbook definition which was examined in Problem 3, definitions in curricular materials often may not help teachers and/or students resolve this conflict. Even the United States' *Common Core State Standards for Mathematics* (Common Core State Standards Initiative, 2010) section on functions echoes the definition equivalence issues encountered in Problem 2. The CCSSM begin this section by declaring that functions "describe situations where one *quantity* determines another" yet, on the same page, they provide an example of a function in which a state name determines its capital city (emphasis added, p. 67).

Many of the PSMTs' difficulties on the three problems may be broadly, and perhaps vaguely, described as a lack of recognition of details or nuance. For example, on Problem 2, only five of the 22 respondents correctly noted that the two definitions of functions were not equivalent because one of them defined the domain and range more restrictively. The results reported herein indicate that many PSMTs may have difficulty with choosing, interpreting, comparing and evaluating definitions, which appear in secondary mathematics curricular materials. It seems likely that these are, at least in part, symptoms of a lack of flexibility and expertise in interpreting and using mathematical definitions. The PSMTs who did not acknowledge that "definite output", in Problem 2, was communicating the same thing as "unique output" (1) did not have the flexibility to make sense of this alternative word-choice, and (2) may not have had the knowledge about definitions to properly assess the equivalence of the two definitions. In sum, pre-service secondary mathematics teachers may benefit from thoughtful modelling of and explicit attention to definition use by teacher educators. This task would be aided by a deeper understanding of how knowledge about mathematical definitions interacts with or is subsumed by subject matter knowledge and pedagogical content knowledge.

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Author

Joshua Chesler, California State University, Long Beach, USA.
Email:<josh.chesler@csulb.edu>