K-8 pre-service teachers' algebraic thinking: Exploring the habit of mind "building rules to represent functions"

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In this study, we examined the ability of grades 1-8 pre-service teachers (PSTs) to engage in thinking about patterns, relationships, and functional rules. Using the algebraic habit of mind Building Rules to Represent Functions as a framework, we examined whether and how well our PSTs (n = 18) used seven features of this habit of mind: organize information, predict patterns, chunk information, use different representations, describe a rule, describe change, and justify a rule. The data revealed distinct and significant differences among our PSTs' demonstrated abilities to justify a rule, organize information, predict a pattern, and describe a rule. Their ability to justify a rule was the weakest of the seven features, and it was correlated with the ability to predict patterns and the ability to chunk information. Our analysis reveals the complexity of the relationships among the seven features and provides a tentative grouping of features that together might support one another. Attention to these relationships has the potential to enhance instruction aimed at helping students learn to analyse problem situations that focus on writing rules to represent functional relationships.

Keywords· algebraic thinking · algebra instruction · teacher knowledge · mathematical knowledge for teaching · pre-service teachers

Background

In recent decades, concerns about students' limited understanding and preparation for the study of more advanced mathematics has stimulated multiple discussions about the role and the nature of school algebra in the K-12 mathematics curriculum. Algebra instruction and students' learning of algebraic concepts and skills have become a focal point in mathematics education research (e.g., Algebra Working Group to the National Council of Teachers of Mathematics, 1997; Britt & Irwine, 2008, 2011; Cai & Knuth, 2011; Irwine & Britt, 2005; Kaput, 1998, 2008; Lacampagne, Blair, & Kaput 1995). The mathematics education community has engaged in discussions about the kind of instruction that has the greatest potential to enable all students to learn algebraic concepts. Today, there is widespread agreement that the study of algebra-based topics should be an integral part of the K-8 mathematics curriculum. Calls for algebra instruction at the K-8 level place emphasis on mathematical experiences that allow K-8 students to "…learn algebra as a set of concepts and competencies tied to the representation of quantitative relationships and as a style of thinking for formalizing patterns, functions, and generalizations" (NCTM, 2000, p. 223). Calls for early algebra stem from the need to provide K-8 students with rich mathematical experiences that originate from, and go beyond, arithmetic and computational fluency. These experiences often come from activities that require students to recognise and articulate the relationships embedded in mathematical situations and examine the underlying structure of identified relationships (Blanton & Kaput, 2011; Moss & McNab, 2011; Britt & Irwine, 2011). A goal of early algebra is to advance students' conceptual knowledge by engaging them in analysing and generalising patterns using multiple representations at early stages of learning mathematics (Carpenter & Levi, 2000; Kieran, 1996; Mulligan, Cavanagh, & Keanan-Brown, 2012; NCTM, 1989, 2000; Silver, 1997).

Research on early algebra instruction (e.g., Blanton & Kaput, 2011; Britt & Irwin, 2011; Cai & Moyer, 2008; Carpenter & Levi, 2000; Kaput, 1998; Kieran, 1996; Radford, 2015; Silver, 1997) shows that early mathematical experiences that create rich connections between the ideas of arithmetic and algebra support students in making a smooth transition from arithmetic to the study of more advanced mathematics. Blanton and Kaput (2003), for example, argued that teachers can provide students with these types of experiences by placing an emphasis on algebraic thinking in their K-8 mathematics classrooms and engaging students in "modelling, exploring, arguing, predicting, conjecturing, and testing their ideas, as well as practicing computational skills" (p. 75).

There has also been research into teachers' understanding of algebraic thinking. Borko et al. (2005) investigated elementary and middle school teachers' broad knowledge of algebraic thinking, including their knowledge of instructional strategies, representations, and curricular materials that foster algebraic thinking in students. They also examined how teachers make sense of student early algebraic experiences. Koellner et al. (2011) examined the algebraic thinking of teachers of middle grades through the lens of instructional strategies teachers use to support their students' algebraic reasoning. More recently, Glassmeyer and Edwards (2016) investigated how teachers of middle grades conceptualise and communicate their understanding of the ideas behind algebraic reasoning and Walkoe (2015) examined how *pre-service* teachers make sense of student algebraic thinking. However, none of these studies assessed how teachers' own mathematical work *demonstrates* their understanding of the ideas essential to algebraic thinking.

In this paper, we focus on grades 1-8 pre-service teachers (PSTs) and examine their demonstrated knowledge of algebraic thinking. It is reasonable to expect that the PSTs' future efforts to foster their students' ability to engage in algebraic thinking will draw not only on their perceptions of algebraic thinking but also on the strategies and ways of thinking they themselves use when solving algebra-based tasks. Therefore, an understanding of PSTs' algebraic thinking, including the strategies they use in solving problems, can support the work of mathematics teacher educators. Specifically, the ultimate purpose of our study is to provide mathematics teacher educators with an understanding of the aspects of algebraic thinking that need to be emphasised, along with the tools to do so, to enhance PSTs' readiness to provide early algebra instruction in their future classrooms.

Algebraic thinking

In the mathematics education literature, the term algebraic thinking has various connotations (e.g., Driscoll, 1999; Kieran & Chalouh, 1993; Lee, 2001; Mason, 1987, 1989; Swafford & Langrall 2000). Often, however, its meaning relates to what Cuoco, Goldenberg, and Mark (1996) described as habits of mind: useful ways of thinking about mathematical content. Swafford and Langrall, for example, interpreted algebraic thinking as the ability to think about unknown quantities as known. Kieran and Chalouh (1993) used this term to describe one's ability to build meaning for the symbols and operations of algebra in terms of arithmetic. Kieran (1996, 2004) refined this interpretation, associating algebraic thinking with the ability to use a variety of representations to analyse quantitative situations in a relational way. Some researchers (e.g., Mulligan & Mitchelmore, 2009; Radford, 2012) emphasised awareness of patterns and structure as core ideas of algebraic thinking. Yet others (e.g., Mason, 1989) drew attention to algebraic abstraction describing the notion of algebraic thinking. Lee (2001) asserted that algebraic thinking hinges on one's ability to reason about a variety of patterns (e.g., numerical, figural) and to detect sameness and differences within them. Lee also defined algebraic thinking in terms of the ability to generalise, to see the general in the particular, to mentally invert and reverse operations, and to think about mathematical relationships rather than mathematical objects themselves. When considering generalising as a major indicator of algebraic thinking, researchers (e.g., Cooper & Warren, 2008; Ellis, 2007; Warren, Cooper, & Lamb, 2006) have drawn attention to the processes involved in the act of constructing general rules from patterns, tables of values, and verbal and abstract representations. In that sense, they interpreted generalisations in terms of both a processes (the activities of generalising, generalising actions) and the products (the general rules resulting from these processes). In addition, Warren, Cooper, and Lamb (2006), among others, discussed the role of generalising in the case of understanding the concept of function. To them, functional thinking involves generalising to understand the relationship between the change of a first variable and the corresponding change of a second variable. They identified three landmark phases of functional thinking: (a) the ability to interpret functions as *actions* that convert inputs to outputs, often using arithmetic calculations; (b) the ability to interpret functions as *processes* that group individual steps into a cohesive repeating sequence of steps; and (c) the ability to internalize functions as *objects* (general rules) having their own characteristics, thus being able to see functions as objects that can be compared with other functions.

The notions of algebraic thinking described above are evident in Driscoll's (1999, 2001) conceptualisation of algebraic thinking. He explained that "facility with algebraic thinking includes being able to think about functions and how they work, and to think about the impact that a system's structure has on calculations" (Driscoll, 1999, p. 1). Driscoll referred to these two aspects of algebraic thinking as habits of mind: Building Rules to Represent Functions and Abstracting from Computations, both linked via the habit of Doing and Undoing (mathematical processes and operations). Kaput (2008) also emphasised two core aspects of algebraic thinking: expressing generalisations using increasingly formal and conventional systems of symbols, and reasoning with symbolic forms. Blanton and Kaput (2011) labelled this first aspect of algebraic thinking as functional thinking, which they conceptualised as "building and generalising patterns and relationships using diverse linguistic and representational tools" (p. 8). They further refined the term algebraic thinking to mean pattern building, conjecturing, generalising, and justifying mathematical relationships between quantities in a way that leads to symbolising and discussing functional relationships.

Building rules to represent functions

Central to the research reported in this article is the first aspect of algebraic thinking as described in both Driscoll (1999) and Kaput (2008), namely what Blanton and Kaput (2011) and Warren, Cooper, and Lamb (2006) termed as functional thinking and Driscoll conceptualised as the habit of mind Building Rules to Represent Functions. That is, in our work with PSTs, we examined the type of algebraic thinking that underlies the ability to generalise numeric patterns and represent them as functional relationships. We conceptualised this aspect of algebraic thinking using Driscoll's (2001) description of the habit of mind Building Rules to Represent Functions as thinking processes. These processes include recognising and analysing patterns, investigating and representing relationships, generalising beyond specific examples, analysing how processes or relationships change, and seeking arguments for how and why rules and procedures work. Specifically, our operational definition of this aspect of algebraic thinking is based on Driscoll's features, which provide characteristics of the processes in which one engages while Building Rules to Represent Functions (Driscoll, 1999, 2001). Throughout this paper, then, we narrow the use of the term algebraic thinking to mean the kind of thinking that results from exercising one or more of the features of the habit of mind Building Rules to Represent Functions (see Table 1).

There are two reasons that we framed our investigation of PSTs' functional thinking using Driscoll's features of the habit of mind Building Rules to Represent Functions. First, while others (e.g., Blanton & Kaput, 2011; Kaput, 2008; Kieran, 1996, 2004; Lee, 2001; Mason, 1989; Warren, Cooper, & Lamb, 2006) also provide considerable insights into functional thinking, their definitions are more elusive. That is, they do not sufficiently explicate or illuminate how one might construct functional rules or expressions of generality. For example, Warren, Cooper, and Lamb (2006) described functional thinking as focusing on one's understanding of relationships between two or more varying quantities, and they distinguished between the processes of generalising and the products of generalising. However, they do not provide clear descriptions of the processes in which one might engage to construct a functional rule. In contrast, Driscoll's features explicate seven processes that characterise functional thinking in a mathematical situation. More specifically, the features he identified provide insights into the specific processes that underlie the act of analysing patterns and relationships in a mathematical situation, and describing them using a functional rule.

Second, as we explain in the section that follows, we also adopted Driscoll's descriptions of the features of the habit of mind Building Rules to Represent Functions for pedagogical reasons. Driscoll and colleagues identified the seven features by observing teachers during their involvement in the Linked Learning in Mathematics Project (Zawojewski & Goldberg, 2000) without identifying how the features might relate to one another. Our goal was to acquire a nuanced understanding of the relationships among the features of Building Rules to Represent Functions in order to help teacher educators plan more effective learning experiences for PSTs.

Table 1
Features of Building Rules to Represent Functions Examined in This Study*

Features	Description of Thinking Exemplified
1. Organizing Information	Ability to organize information in ways useful for uncovering patterns, relationships, and the rules that define them
2. Predicting Patterns	Ability to discover and make sense of regularities in a given situation
3. Chunking Information	Ability to look for repeating bits in information that reveal how a pattern works

4. Different Representations	Ability to think about and try different representations of the problem to uncover different information about the problem
5. Describing a Rule	Ability to describe steps of a procedure or a rule explicitly or recursively without specific inputs
6. Describing Change	Ability to describe change in a process or a relationship explicitly as a functional relationship between variables
7. Justifying a Rule	Ability to justify why a rule works for any input

* Adapted from Driscoll (2001)

Teacher knowledge and teacher preparation

Teachers are central to effecting the educational reform related to early algebra instruction, but their own experiences with traditional school algebra often influence their views of algebraic thinking, limiting or even countering their efforts. Thus, the call for the early introduction of algebraic ideas has many inherent challenges. Research shows that both practising teachers' and PSTs' understanding of algebraic topics often consists of a fragmented knowledge of a disconnected system of symbols and procedures (Ball, 1990; Ma, 1999, van Dooren, Verschaffel, & Onghena, 2002). Teaching that is informed by such limited knowledge belies the goals of early algebra instruction. Warren (2008) argued that to embrace the focus on early algebraic ideas in the elementary and middle school classrooms, teachers need to "reconceptualise arithmetic and patterning, drawing out structures in an activity based environment" (p. 30). Effective early algebra instruction requires the careful preparation of elementary and middle school teachers. To take advantage of opportunities to engage students in algebraic thinking, teachers need to understand the ideas on which algebraic thinking hinges (National Mathematics Advisory Panel, 2008; Glassmeyer & Edwards, 2016; Greenberg & Walsh, 2008). As Blanton and Kaput (2003) summarise, "Elementary teachers need their own experiences with a richer and more connected algebra and an understanding of how to build these opportunities for their students," (p. 70). To effectively prepare students for success in algebra, teachers must have a robust ability to think algebraically themselves (Magiera, van den Kieboom, & Moyer, 2010, 2011, 2013; van den Kieboom, Magiera, & Moyer, 2014).

One way to begin strengthening teachers' algebraic thinking is to strengthen the preparation of PSTs. Recent research efforts provide some direction for the preparation of PSTs. Hill (2010) argued that effective teacher preparation programs should be deeply grounded in an understanding of the specialised content and pedagogical knowledge needed for teaching. She made the case for a research agenda that provides a fine-grained understanding of the type of knowledge that teachers need to be successful in their work. The research reported in this paper responds to this need by seeking a nuanced understanding of the aspects of PSTs' thinking that support their ability to build rules to represent functions.

Data for this paper comes from a larger project in which we investigated how teacher educators can assess and strengthen the various aspects of PSTs' knowledge of algebraic thinking. In this paper, we focus our discussion on the ability of PSTs to use different features of the habit of mind Building Rules to Represent Functions. To provide valuable insights for teacher educators about ways to enhance the ability of PSTs to analyse mathematical situations with a focus on building rules to represent functions, we sought to answer the question: *Which features of the habit of mind Building Rules to Represent Functions appear to support and strengthen one another in our PSTs' written solutions to algebra-based problems?*

Method

Participants and setting

Featured as participants in this report are 18 undergraduate PSTs (juniors and seniors) at a large private Midwestern university in the U.S. Sixteen of the PSTs were female and two were male. The seniors were in their final academic semester prior to student teaching. All PSTs were enrolled in a mathematics content course integrated with an education field experience course. The content course, taught in the Mathematics Department, was the last in a conceptually-based 3-course sequence in mathematics for grades 1-8 education majors. The course was designed to build PSTs' awareness of the foundational features upon which algebraic thinking rests and to develop their ability to compare, connect, and generalise across multiple algebra topics within the 1-8 mathematics curriculum. Rather than explicitly teach PSTs how to engage in thinking consistent with each feature, we engaged them in activities to draw their attention to their spontaneous use of features of the Building Rules to Represent Functions habit of mind. To do so, we used algebra-based activities that solicited multiple solutions and representations of algebra-based ideas, and we provided opportunities for the PSTs to share, explain, compare, and interpret various representations and reasoning. To foster the PSTs' awareness of the ideas behind algebraic thinking and to help them develop a language for describing algebraic thinking, the mathematics course instructor explicitly discussed the features of the habit Building Rules to Represent Functions. The PSTs were asked to reflect on these ideas and demonstrate the use of algebraic habits of mind in their own work. To promote the PSTs' diagnostic abilities, we also engaged them in analysing samples of middle school students' written work for evidence of thinking consistent with the habit of mind Building Rules to Represent Functions. To further strengthen their ability, the PSTs were asked to conduct two problem-based interviews of a middle school student from their pedagogy field placement, and conduct an analysis of "their" student's responses with a focus on that student's ability to demonstrate features of the habit of Building Rules to Represent Functions.

Data sources and data collection

To provide a robust measure of the PSTs' algebraic thinking in a variety of problem-solving situations that foster thinking about functional relationships, we analysed their written solutions to 125 problems, which they completed for individual homework assignments and during our in-class formative and summative assessment activities. The problems were designed to elicit the PSTs' thinking consistent with the seven features of the habit Building Rules to Represent Functions (Table 1). Our goal was to identify how well PSTs are able to use each of the seven features to solve problems that require the habit of mind Building Rules to Represent Functions.

Data analysis

We initiated the data analysis processes by identifying the specific features of algebraic thinking (AT) encouraged by each of the 125 problems embedded in the tasks[1] solved by our PSTs over the semester. To do so, we developed a set of seven feature-specific criteria and used it to determine the features of algebraic thinking elicited or encouraged by each task. The Task Analysis section below provides the seven criteria and examples of their application. The three authors and a trained research assistant independently applied the feature-specific criteria to a subset of tasks and negotiated their independent interpretations until a 100% agreement on the features encouraged by each task was reached. This process was iterative. Driven by the need to establish the validity and reliability of our process, we used each of the task negotiations to help confirm or modify our seven feature-identification criteria.

After we had identified the features elicited or encouraged by each task, we developed a scoring rubric for assessing the quality of the pre-service teachers' written use of the identified features to solve each task (see Appendix A). Then, the three authors and a trained research assistant independently scored a randomly selected subset of tasks. We compared independent results and cited specific examples to clarify the scoring rubric and negotiate scoring agreement to 100%. Then, for each PST, we used the scoring rubric to rate each of his or her written solutions on how well the solution utilised each identified AT feature.

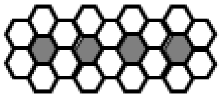
Using the feature ratings for each PST on each task, we quantified each PST's ability to use each AT feature (AT-feature score) by averaging his or her ratings on each of the seven features across the collection of tasks. This resulted in seven AT-feature scores for each PST. We also computed the average of the seven AT-feature scores to provide a measure of each PST's overall ability to use algebraic thinking (AT-composite score). We then used the Friedman and Wilcoxon tests, to test the hypothesis that the PSTs' ability to build rules to represent functions did not differ among the seven features. We decided to use the Friedman and Wilcoxon tests for three reasons. First, they allowed us to compare the PSTs' performance on all seven features simultaneously and pairwise. Secondly, because the non-parametric Friedman test is less sensitive to sample size and the assumption of normality than its parametric alternative, repeated measures ANOVA test. Third, we selected the Friedman test because its results are customarily reported together with information about the effect size giving insight into practical significance of our work. In other words, from the perspective of teacher educators we were not only interested in knowing whether or not the PSTs' ability to build rules to represent functions varied across the seven features, but we were also interested in understanding the magnitude of any statistically significant differences our analysis would reveal. Furthermore, to provide an answer to our research question we also conducted a correlation analysis to determine which features of algebraic thinking might support each other and which appear to be unrelated to one another. In the remainder of this Methods section, we illustrate our task analysis and scoring processes.

Task analysis

Consider the two problems that constitute the Flower Beds task presented in Figure 1.

[1]As expected, not all tasks elicited or encouraged all seven features. Accordingly, we rated the PSTs' solutions only on the features that were explicitly elicited or encouraged by the 125 individual problems that comprised the task statements (78 tasks in all).

Flower Beds[a]



The city council wishes to create 100 flower beds and surround them with hexagonal paving slabs according to the pattern shown above. (In this pattern 18 slabs surround 4 flower beds.)

(1) How many slabs will the council need?
(2) Find a formula that the council can use to decide the number of slabs needed for any number of flower beds.

[a]Shell Centre for Mathematical Education, 1984, p. 64.

Figure 1. Example of the tasks used in the mathematics content course (See Appendix B for examples of all task types).

We judged that a task elicits or encourages the use of *different representations* if a problem solver could benefit from using different representations to organize information about the task situation to generate a task solution. For example, we believed that the Flower Beds task encourages thinking about different representations because using different representations (e.g. numerical, verbal, symbolic) is a useful strategy for organizing one's observations about the sequence of the flower beds to make predictions about the collection of flower beds as a whole.

We judged that a task elicits or encourages *organizing information* if the task solution requires keeping track of information needed to help PSTs make sense of the problem situation. For example, we believed that both problems within the Flower Beds task would elicit and encourage organizing information because in order to uncover a pattern and develop a general rule, PSTs would have to determine the structure of the sequence of flower beds. To do so, they would have to keep track of the information about both the changing and invariant aspects of that sequence.

We judged that a task elicits or encourages *chunking information* if the task solution requires identification of repeating pieces of information that help to describe how the pattern works. To help predict the Flower Beds pattern, for example, some PSTs would recognize that every flower bed after the first would require adding four additional slabs. Other PSTs might, on the other hand, recognize that the diagram's chunk of four flower beds could be used 25 times to create 100 flower beds. To correctly predict the flower bed pattern (see criterion below), however, the latter PSTs would have to realize that every chunk of four flower beds shares two overlapping slabs with the adjacent chunk.

We judged that a task elicits or encourages *predicting a pattern* if the task solution requires that PSTs examine information to uncover regularities across cases. Problems that elicit or encourage predicting patterns demand the development of an understanding of cases that are not present, and thus necessitate thinking about a change across pattern entries. For example, we believed that the Flower Beds task fosters thinking about patterns because the statement of the Flower Beds task prompted the PSTs to think about how 5 flower beds differs from 4 flower beds, and more generally, how any number *n* of flower beds differs from *n-1* flower beds.

Extending the flower bed sequence to answer problem (1) within the Flower Beds task requires that the PSTs consider how the flower beds pattern works, and thus engage in predicting a pattern.

We judged that a task elicits or encourages *describing a rule* (or the steps of a procedure) if the task either directly calls for developing a general rule or if the development of the rule is fostered by the need to provide general information about specific inputs. For example, both problems within the Flower Beds task elicit thinking about a rule. In Problem (1) finding the total number of slabs needed for 100 flower beds encourages the PSTs to develop an alternative for adding 4 slabs 99 times to the 6 slabs required for the first flower bed. Problem (2) directly calls for a *general* rule that allows the PST to predict the total number of slabs for any sequence of flower beds.

We judged that a task elicits or encourages thinking about *describing change* if the task solution requires recognition of the functional relationship between independent and dependent variables. That is, PSTs would have to describe the net change in the input variable in relationship to the net change in the output variable. We did not believe that the Flower Beds task would necessarily elicit or encourage describing change in the total number of slabs as a function of change in the number of flower beds even though the task provided the opportunity to do so.

Finally, we judged that a task elicits or encourages *justifying a rule* if a problem statement explicitly calls for justifying *why* the rule or procedure works. Because the Flower Beds task did not directly call for providing justification for *why* the developed rule gives the correct total number of slabs for *any* number of flower beds, we did not believe the task would necessarily engage PSTs in justifying their conjectures about the validity of the rule they developed.

AT scoring rubric

We rated each PST's demonstrated use of a feature of algebraic thinking as (3) *proficient*, (2) *emerging*, or (1) *not evident* according to the following AT scoring rubric:

Proficient. We rated a PST's use of an identified feature of algebraic thinking as (3) proficient if the problem elicited or encouraged the use of the feature, if the written solution revealed the PST's thinking characteristic of that feature, if the feature was carried out correctly, and if the use of the feature was linked directly to the context of the problem. We placed an emphasis on PSTs' use of each feature in context because to effectively model thinking consistent with each feature while working with their future students, PSTs will need the ability to articulate the meaning of each feature within a specific problem context. For example, we rated a PST's use of the Justifying a Rule feature as (3) proficient if the solution showed evidence that the pre-service teacher correctly used evidence based on an abstract numeric pattern to justify why the stated rule works for any number and the justification was directly linked to the context of the problem.

Emerging. We rated a PST's use of an identified feature of algebraic thinking as (2) emerging if the problem elicited or encouraged the use of the feature, if the written solution documented the PST's thinking characteristic of that feature, and if the feature was carried out correctly, but without direct links to the context of the problem. For example, we rated a PST's use of the Justifying a Rule feature to solve a contextually-based problem as (2) emerging if the solution documented that the PST correctly used evidence based on an abstract numeric pattern to justify why the stated rule works for any number, but the justification was based on the abstract numeric pattern gleaned from a table or sequence of numbers rather than directly linked to the context of the problem. We also rated a PST's use of an identified feature of algebraic thinking as (2) emerging if the written solution articulated the PST's thinking characteristic of that feature with direct links to the context of the problem, but was carried out incorrectly. For example, we rated a PST's use of the Justifying a Rule feature as (2) emerging if the solution provided evidence linked to the problem to justify why a rule works for any number, but the justification was incorrect.

Not evident. Finally, we rated the strength of a PST's thinking as (1) not evident on an identified feature if the problem elicited or encouraged the use of the feature, but the PSTs' written solution did not show evidence of the thinking characteristic of that feature. For example, we rated a PST's use of the Justifying a Rule feature as (1) not evident if the problem statement elicited the justification of a rule, but the PST only used a rule to compute answers for specific cases, rather than attempt to argue that the rule could be used to compute the answer for any case. A detailed rubric for all seven features is included in Appendix A.

Our use of 3-point scoring rubrics to assess PSTs' performance is similar to the practice of other mathematics education researchers (e.g., Jacobs, Lamb, & Philipp, 2010). We realise that a scale with a greater number of points would potentially afford a greater variance from which to distinguish levels of proficiency, however, we chose a 3-point scale for reasons of greater reliability and validity. For example, a 3-point scale clearly allows for greater reliability of scoring than any finer-grained scale with a greater number of possible points. Regarding validity, if we subdivided our "(2) emerging" rating to reflect a variety of behaviours, we would not be able to validly justify which of the behaviours was more advanced than the others. That is, we would be unable to rank the various emerging behaviours on each feature into an ordinal scale, thus forcing the use of a nominal scale and making our analysis unwieldy.

We now use the Flower Beds task presented in Figure 1 to serve as the context for presenting specific examples of how we rated the PSTs' competencies with respect to each feature of algebraic thinking examined in this research. Figures 2 and 3 include examples of solutions and explanations that the PSTs in our study (PST #10 and PST #9) provided for the Flower Beds task. We selected these examples not only to illustrate the variety of the solution approaches, but also because both show evidence of the use of each of the seven features, providing us with the opportunity to illustrate our assessment of the strength of the PSTs' thinking with respect to all features.

Pre-service teacher #10's solution

Figure 2 shows PST#10's solution to the Flower Beds task.

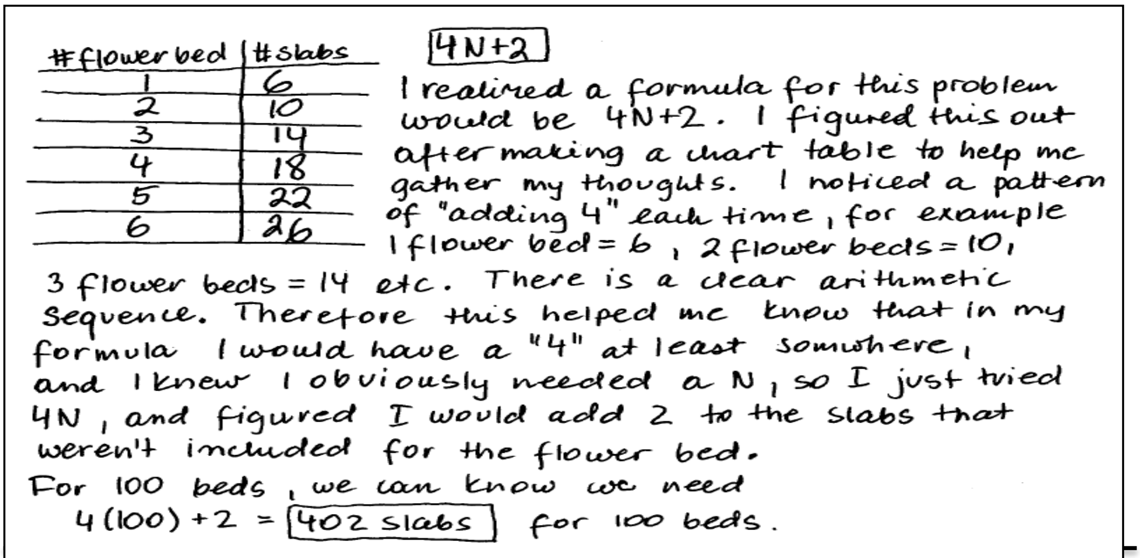


Figure 2. Flower Beds task, PST #10.

As explained in the previous section, although the problems within the Flower Beds task do not elicit or encourage all seven features, in the case of PST #10, her solution shows that she engaged in thinking consistent with all seven features. For that reason, we selected PST #10's solution to illustrate our scoring system, even though our data analysis only utilised scores for the features encouraged in the statement of the problem. In this way, each PST was rated on the same opportunities to demonstrate facility with the seven features.

Proficient ratings. We rated PST #10's solution (3) proficient on each of the following AT features: Feature 1 (Organizing Information), Feature 2 (Predicting Patterns), Feature 3 (Chunking Information), Feature 4 (Different Representations), and Feature 5 (Describing a Rule). Her solution demonstrates proficiency using these features in the following ways: Feature 1 (Organizing Information) because her use of a table demonstrates her ability to organise information in ways useful for uncovering patterns, relationships, and the rules that define them; Feature 2 (Predicting Patterns) because her reference to a pattern of "adding 4" demonstrates her ability to discover regularities in a given situation. We rated PST #10's performance on Feature 3 (Chunking Information) as (3) proficient because she provided a salient example ("…1 flower bed=6 [slabs], 2 flower beds=10 [slabs], 3 flower beds=14 [slabs] etc.") to clarify her description of the pattern of adding 4, which shows her ability to look for repeating bits of information that reveal how a pattern works. We also rated her use of Feature 4 (Different Representations) as (3) proficient because of her appropriate use of tabular, algebraic, and verbal representations. Finally, we rated her as (3) proficient on Feature 5 (Describing a Rule), because her use of the expression 4N + 2 demonstrates her ability to describe a rule explicitly without specific inputs.

Emerging ratings. If the Flower Beds task had asked for a justification of the rule, we would have rated the strength of PST #10's performance on Feature 7 (Justify a Rule), as (2) emerging. Although the Flower Beds task did not directly encourage the feature Justify a Rule, we infer that PST #10 was motivated to do so because justification was a ubiquitous component of classroom discussions. Her explanation of the thought process she used to describe her formula reveals that she had little understanding of why her rule worked. Although she used the pattern "…of adding 4 each time," to devise her rule, she utilised a guess-and-check strategy to do so. That is, in her explanation of her reasoning she did not adequately link the 4N or the 2 of her rule to the problem situation. Specifically, her statement, "…this helped me know that in my formula I would need a '4' at least somewhere, and I knew I obviously needed an N, so I just tried 4N, and figured I would add 2 to the slabs that weren't included in the flower bed," shows that her rule is motivated by a little knowledge and quite a bit of guessing. From this we infer that she had at best a partial understanding that the rule [S =] 4N + 2 indicates that each additional flower bed requires the addition of four new slabs to the two that will be shared.

Not evident ratings. PST #10's description of the thought process she used to describe her formula provides evidence that she was not explicitly thinking of the changes in the number of slabs as a function of changes in the number of flower beds. Specifically, had the Flower Beds task directly encouraged a description of change we would have rated PST #10's proficiency using Feature 6 (Describing Change), as (1) not evident because neither the table nor her verbal explanation contained any evidence that she explicitly considered a net change in the total number of slabs as a function of a net change in the total number of the flower beds.

Pre-service teacher #9's solution

PST #9's solution to the Flower Beds task is shown in Figure 3. Like PST #10, PST #9's solution provides us with sufficient documentation to rate her performance on all seven features. Although her solution is quite sophisticated in many respects, it nonetheless reveals a misunderstanding of the concept of proportionality which affects her score on Feature 2 (Predicting a Pattern).

Proficient ratings. We rated PST #9's use of Features 1 and 3 through 7 as (3) proficient. Despite her misuse of proportionality (discussed below), PST #9's use of verbal descriptions, a formula, a table, and a diagram in her solution reveals an ability to create different representations (Feature 4) to guide her thinking about the characteristics of the flower beds design. PST #9's organization of the problem information (Feature 1) enabled her to focus on repeating bits of information (Feature 3) in at least several different ways. First, her diagram and her table revealed the regularity of consistently adding four slabs for every additional flower bed. Secondly, her diagram shows that a grouping of two slabs is shared by each pair of adjacent flower beds. Third, the diagram shows how the two types of information are used in her rule, $6F - [(F - 1) \times 2] = S$. Specifically, the diagram shows that the total number of slabs (S) needed for (*F*) flower beds can be determined by subtracting F - 1 groups of 2 overlapping slabs from the F groups of 6 slabs that are ostensibly required for *F* flower beds.

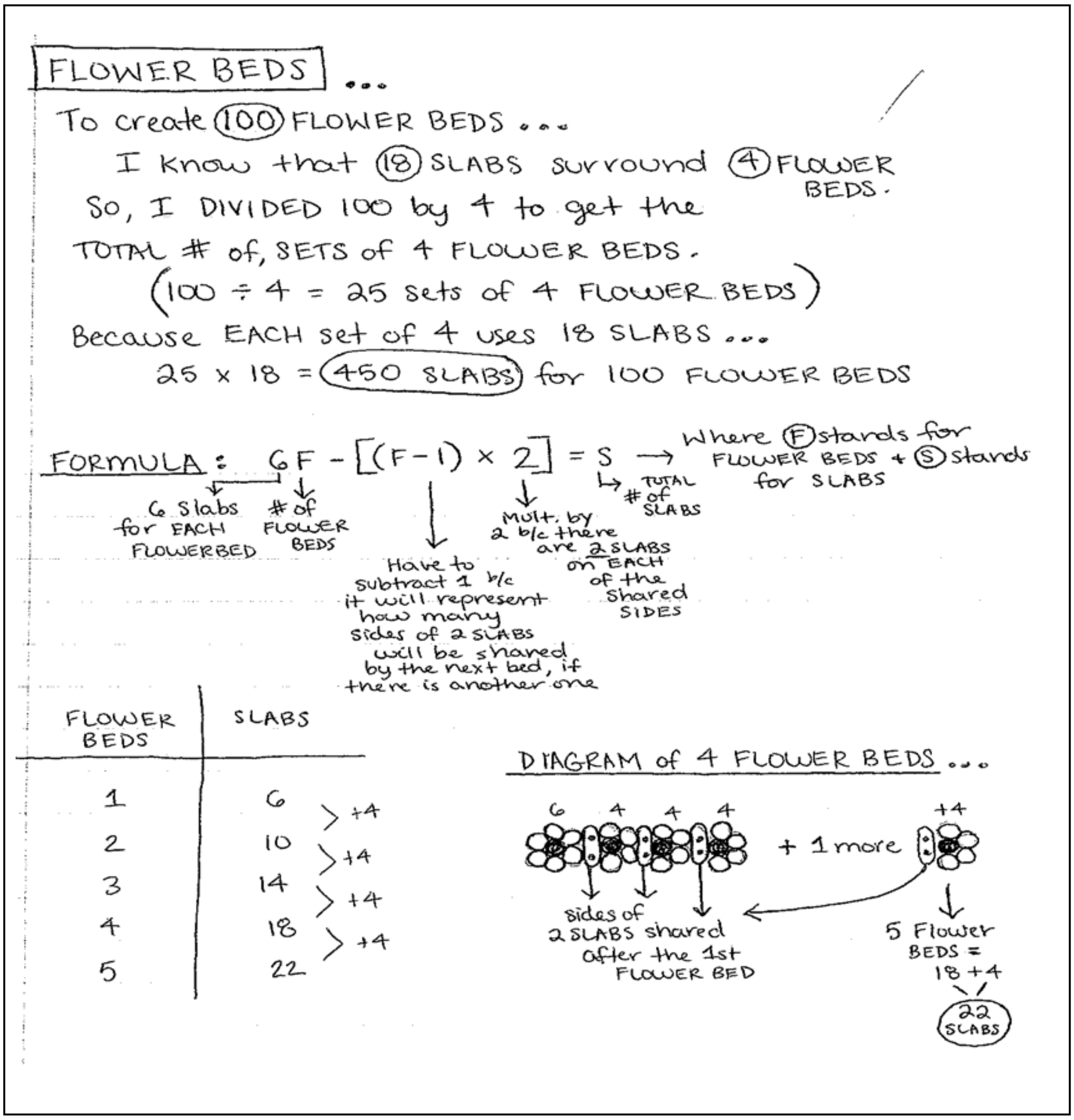


Figure 3. Flower Beds task, PST #9.

PST #9's solution also demonstrates the use of Feature 6 (Describing Change). Her table and diagram provided evidence of appropriate thinking about the change (+4) in the total number of slabs that corresponds to each unit change (+1) in the total number of flower beds. This is in clear contrast to our (1) not evident rating of PST #10's ability to describe change. The difference in the two solutions is clearly identifiable. Even though PST #10 organised the flower beds information in a table similar to PST #9's, PST #9 exhibited clear evidence of understanding the functional relationship between a change in the number of flower beds ("+1 more" in her diagram) and the resulting change in the number of needed slabs ("+4" above the 5th flower bed in her diagram), while PST #10 did not.

Despite PST #9's incorrect solution to problem (1), we also rated her (3) proficient on Feature 7 (Justifying a Rule), and on Feature 5 (Describing a Rule). We did so for two reasons. First, her solution to problem (2) clearly demonstrates an insight and ability to describe and justify an appropriate "formula" (rule), namely $6F - [(F - 1) \times 2] = S$. Second, her answer to problem (1) (450 slabs) was incorrect because she computed it using a rule that was based on her incorrect prediction that the pattern was proportional (see below). In fact, her subsequent description and justification of the rule in problem (1) would have been correct if the pattern had actually been proportional. Therefore, we rated PST #9 as being (3) proficient on Features 7 (Justifying a Rule) and 5 (Describing a Rule) because we believe that her incorrect solution in problem (1) is due to her inability to reliably predict a pattern, rather than to her inability to justify or describe a rule.

Emerging rating. We rated PST #9's solution on Feature 2 (Predicting a Pattern) as (2) emerging. The progression of PST #9's solution indicates that she may have predicted and applied an incorrect pattern to answer problem (1) before she thoroughly analysed problem (2). In particular, our rating reflects her inability to reliably predict the correct pattern. She inappropriately used proportionality as the pattern for problem (1) of the Flower Beds task (100 flower beds: 4 flower beds = x slabs : 18 slabs). On the other hand, her formula for problem (2) was correct because the pattern she predicted was based on her correct understanding of chunking information and describing change. Note that in her solution of problem (2) she did not use her formula to generate the answer to problem (1), which would have yielded $6 \times 100-[(100-1) \times 2] = 402$. We believe that her failure to do so helped mask the disparity between the incorrect pattern she predicted in her answer to problem (1) and the correct pattern she predicted in her answer to problem (2). Because she did not notice and correct the inconsistencies between the patterns she identified in problems (1) and (2), we assigned a rating of (2) *emerging* on predicting a pattern for the Flower Beds task as a whole.

Results

Strength of algebraic thinking

Presented in Table 2 are the seven medians of the 18 PSTs' AT feature scores. A Friedman test was conducted to compare the PSTs' medians across all seven features. The test was significant $\chi^2(6, N = 18) = 25.95$, $p < 0.001$, and the Kendall's coefficient of concordance of 0.24 indicated fairly strong differences in PSTs' performance on the seven features. Of all the medians, the median on the Justifying a Rule feature (2.214) was the smallest. Follow-up pairwise comparisons were conducted using a Wilcoxon test with the Bonferroni adjustment for multiple comparisons to control, at the 0.05 level, for the Type I errors across the set of comparisons. The test confirmed significant differences between the Justifying a Rule medians and three other medians: Organizing Information ($z = -3.549$, $p < 0.001$), Predicting a Pattern ($z = -3.332$, $p < 0.001$), and Describing a Rule ($z = -3.332$, $p < 0.001$). In each case, the effect size was large: $r = 0.837$, $r = 0.785$, $r = 0.785$ respectively. The other differences were not statistically significant.

Table 2
PSTs' Median AT-Feature Scores

	Organizing Information (n=18)	Predicting Patterns (n=18)	Chunking Information (n=18)	Different Representations (n=18)	Describing a Rule (n=18)	Describing Change (n=18)	Justifying a Rule (n=18)
#(%) of Eliciting	78 (62%)	58 (46%)	38 (30%)	27 (22%)	66 (53%)	71 (57%)	40 (32%)

Problems							
Median	2.564	2.478	2.404	2.617	2.594	2.567	2.214

Note that each AT-feature score was the mean of all the scores that an individual PST received on a given feature, e.g. the mean of 40 scores of 1, 2, or 3 on Feature 7 (Justifying a Rule). As a result, in our Friedman comparison, a PST that scored all 2s would be considered the same as a PST who scored half 3s and half 1s. Although the performance of the two PSTs was not identical, it would be reasonable to classify both of their abilities to Justify a Rule as (2) emerging. It is emerging in the first PST because he or she is able to justify rules proficiently in only half the problems. It is emerging in the second PST because he or she is able to partially justify rules in all the problems.

We generated an AT-composite score for each PST by averaging his/her seven AT-feature scores. The AT-composite score rated a PST's overall ability to think algebraically (as defined by our operational definition of algebraic thinking). The mean of the 18 AT-composite scores was $M = 2.455$ (max 3); SD = 0.243, which indicates that our PSTs demonstrated a rather high overall algebraic thinking ability.

Associations among the features of algebraic thinking

To uncover which features of PSTs' own algebraic thinking might support and strengthen one another, we conducted Pearson correlations between all 21 pairs of the seven AT-feature scores. The results are summarised in Table 3.

Table 3
PSTs' Mean AT-Feature Scores

AT Feature	Feature 1	Feature 2	Feature 3	Feature 4	Feature 5	Feature 6	Feature 7
1. Organizing Information	-						
2. Predicting Patterns	0.717**	-					
3. Chunking Information	0.535*	0.914**	-				
4. Different Representations	0.387	0.466	0.399	-			
5. Describing a Rule	0.508*	0.771**	0.727**	0.277	-		
6. Describing Change	0.462	0.337	0.321	0.122	0.173	-	
7. Justifying a Rule	0.444	0.537*	0.484*	0.324	0.360	0.376	-

*$p < 0.05$, ** $p < 0.01$

Eight of the correlations were significant and 13 were not. Interestingly, none of the 11 correlations that involve Feature 4 (Different Representations) or Feature 6 (Describing Change) were statistically different from zero. The two remaining non-significant correlations were between Features 7 (Justifying a Rule) and 1 (Organizing Information), and between Features 7 (Justifying a Rule) and 5 (Describing a Rule).

Figure 4 illustrates the relative strengths of the eight significant correlations. The heavier weights of four segments in the diagram illustrate that those four significant pairwise correlations are greater ($0.717 \leq r \leq 0.914$) than the other four significant correlations ($0.484 \leq r \leq 0.537$). These results show that all pairwise correlations between the AT-feature scores for Features 1 (Organizing Information), 2 (Predicting Patterns), 3 (Chunking Information), and 5 (Describing a Rule) were significant ($p < 0.05$). These correlations indicate that Features 1, 2, 3, 5 (and to some extent 7) may support and strengthen one another. On the other hand, neither Feature 4 (Different Representations) nor 6 (Describing Change) was associated with the ability to use any of the remaining five features of algebraic thinking or with one another. This indicates that the AT processes that these two features represent may have been employed independently of the other five features and each other.

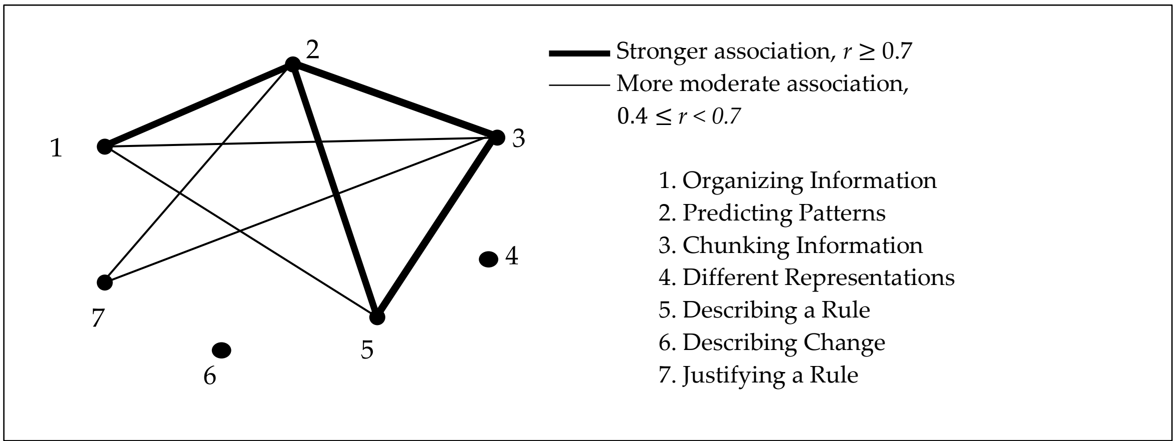


Figure 4. Map of relative strength of pairwise correlations among seven features of algebraic thinking.

Limitations of the findings

Before we present the conclusions we have drawn from our analysis, it is necessary to recognise the limitations of our research. First, the small number of study participants (PSTs), lack of comparison groups, and absence of other types of courses or settings necessitate that our results be interpreted with caution. Secondly, the PSTs in our study were all students in a private university, which might also limit the generalisability of our results. Finally, the generalisability of our results might also be limited given our choice to rely on written solutions as the sole measure of our PSTs' algebraic thinking. From one perspective, this choice might be less problematic considering Pugalee's (2001) findings, which revealed that the strategies students use in their written solutions do not differ greatly from the strategies they use in "think out loud" protocols. Thus, our reliance on PSTs' written responses might actually serve to strengthen the potential contributions of our work to teacher preparation. It is reasonable to expect that the PSTs will extend their written descriptions of the features of the habit of mind Building Rules to Represent Functions when they work with their future students. That is, they will likely draw on the strategies they themselves articulate in their written work, while fostering thinking about building rules to represent functions in their future students. From another perspective, however, given that the framework used in our study characterises the processes in which one engages to build rules to represent functional relationships, it is possible that some of the processes might be easier, and some more difficult, to clearly articulate in writing. Therefore, focusing on only written measures could, to some degree, obscure information about our PSTs' abilities to use each feature of algebraic thinking. Of course, a similar limitation may apply to processes articulated by PSTs during "think out loud" interviews.

Given these limitations, however, because we used a large collection of tasks that the PSTs solved in a variety of situations (e.g., homework, formative and summative assessment activities throughout the course), and because across the collection of tasks, PSTs had numerous opportunities to demonstrate their use of all features of algebraic thinking, we believe that our written measures adequately reflect the PSTs' *overall* abilities to employ the different features of algebraic thinking. We also believe that our decision to draw on a large collection of written tasks administered across the entire semester allowed us to provide a more stable assessment of the PSTs' ability to engage in each of the seven features of interest.

Discussion

Hill (2010) addressed the need for a fine-grained analysis of highly specialised teacher content and pedagogical knowledge. She made the case that the effectiveness of teacher preparation programs is dependent on a strong understanding of how highly specialised aspects of mathematics content and pedagogical knowledge interrelate. Our inquiry provides a fine-grained analysis of one aspect of K-8 PSTs' algebraic thinking, and our results provide some insights for teacher educators to consider as they prepare PSTs for the challenges of teaching K-8 mathematics with a focus on algebraic thinking. Our results augment the findings of other researchers (e.g. Borko et al., 2005; Glassmeyer & Edwards, 2016; Koellner et al., 2011; Walkoe, 2015) by assessing how middle grades PSTs' own mathematical work *demonstrates* their understanding of the ideas essential to algebraic thinking.

First, we found that our PSTs had relatively high overall AT abilities and were able to proficiently use many features of the habit of mind Building Rules to Represent Functions to solve algebra-related problems. Consistent with Castro (2004) and Morris (2010), however, our PSTs' ability to justify a rule or procedure was weak in comparison to their ability to employ other features of the algebraic habit of mind Building Rules to Represent Functions. Our analysis of the PSTs' work revealed the complexity of the relationships among the seven features that support thinking about rules representing functions. Taken together, these relationships suggest that the abilities to organize information, predict patterns, chunk information, and describe a rule may support one another in a mutual, symbiotic, and holistic way. But, the ability to justify a rule appears to be related only to the Predicting Patterns and Chunking Information features of Building Rules to Represent Functions. The ability to proficiently use the Different Representations feature appears not to be related to any of the six other features.

These results suggest important directions that mathematics teacher educators might consider taking. The PSTs' relatively poor proficiency on the Justifying a Rule feature indicates that strengthening PSTs' ability to justify may need to be a targeted focus of the entire teacher education curriculum, rather than a single semester devoted to algebraic thinking. The uncovered relationships among the seven features suggest that mathematics teacher educators might help PSTs strengthen their algebraic thinking by targeting learning activities at appropriate groups of features. In particular, the uncovered relationships suggest that activities requiring PSTs to justify rules by reflecting on patterns and chunks of information that describe them may provide an effective way for the Justifying a Rule feature to develop into a useful habit of mind. Furthermore, our results suggest that planning learning experiences that concurrently engage PSTs in predicting patterns, describing a rule, organizing and chunking information may be worthwhile to support PSTs' facility in using the AT habit of Building Rules to Represent Functions. On the other hand, learning experiences devoted to helping PSTs become proficient at using different representations or describing change may not benefit from synchronisation with the other features.

Algebraic thinking is at the heart of teaching and learning in K-8 mathematics classrooms. Supporting PSTs' own algebraic thinking ability should be an important goal for elementary and middle school mathematics teacher educators. Our study provides a window into the complexity of PSTs' thinking that underlies the habit of Building Rules to Represent Functions. The results can help mathematics teacher educators and mathematics education researchers design teacher education programs sensitive to important issues related to early algebra instruction. Despite the limitations stated earlier, we believe that our results point toward promising avenues for mathematics teacher educators to pursue. These results underscore the importance of strengthening PSTs' facility with algebraic thinking, with a focus on their ability to justify algebraic rules and procedures. Future research can further examine the relationships among the features of the habit of mind Building Rules to Represent Functions using other measures (e.g., problem based interviews) of PSTs' abilities. Focusing on other habits of mind essential to algebraic thinking (e.g., Abstracting from Computations, and Doing and Undoing mathematical processes and operations) can also create a more comprehensive understanding of PSTs' facility with algebraic thinking processes. Concurrently focusing on the three habits can enhance the usefulness of the findings presented here, creating more comprehensive directions for mathematics teacher educators to strengthen PSTs' facility with these processes.

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Appendix A

Scoring rubric for assessing pre-service teachers' use of the seven features of Building Rules to Represent Functions

	Proficient (3)	Emerging (2)	Not Evident (1)
Organizing Information	The PST organized the information in the problem in a way that is useful for discovering underlying patterns and relationships; AND, the organizational scheme used is explicitly connected to the context of the problem (e.g., uses a table to organize information in the problem and clearly relates table entries to the context of the problem).	The solution indicates that the PST organized the information in the problem in a way that is useful for discovering the underlying patterns and relationships; BUT, the organizational scheme used is not explicitly connected to the context of the problem (e.g., problem information is organized in a table but table entries are not contextualized, i.e., their meaning explained with links to the context of the problem).	The solution does not indicate that the PST organized the information in the problem in a way that is useful for discovering the underlying patterns and relationships
Predicting Patterns	The PST shows understanding of how the pattern works (e.g., terms beyond the perceptual field are identified correctly, explicit or recursive rule that describes the pattern is correct); AND, the pattern or discovered regularities are explicitly connected to the context of the problem.	The solution indicates PST's understanding of how the pattern works (e.g., terms beyond the perceptual field are identified correctly, explicit or recursive rule that describes the pattern is correct); BUT, the pattern or discovered regularities are not explicitly connected to the context of the problem.	The solution does not indicate the PST's understanding of how the pattern works; OR the pattern is identified incorrectly.
Chunking Information	The PST identified repeated chunks of information that explain how the pattern works; AND, the identified repeated chunks of information are explicitly connected to the context of the problem.	The solution indicates that the PST identified repeated chunks of information to explain how the pattern works; BUT, the identified repeated chunks are not explicitly connected to the context of the problem.	The solution does not indicate that the PST identified repeated chunks of information that explain how the pattern works, OR repeated chunks of information in the pattern are identified incorrectly.
Describe a Rule	The PST described the rule (verbal or symbolic) to represent the uncovered	The solution indicates that the PST described the rule (verbal or symbolic) to represent	The solution does not indicate that the PST identified and described the steps of a

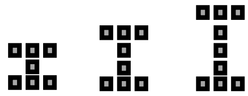
	relationship; AND, the rule is explicitly connected to the context of the problem.	the uncovered relationship; BUT the rule is not explicitly connected to the context of the problem.	rule through which the relationship embedded in the problem can be represented.
Different Representations	The PST used different representations (e.g., verbal, numerical graphical, or algebraic) to uncover, and explore information embedded in the problem; AND, the representations used are explicitly connected to the context of the problem.	The solution indicates that the PST used different representations (e.g., verbal, numerical graphical, algebraic) to explore information embedded in the problem; BUT, the representations used are not explicitly connected to the context of the problem (e.g., uses a list of numbers without contextualising their meaning).	The solution does not indicate that the PST used different verbal, numerical, graphical, or algebraic representations to uncover different information about the problem.
Describing Change	The solution indicates that the PST recognized the change in a process or relationship as a function of the relationship between variables in the problem (e.g.., change in the input variable with respect to the change in the output variable); AND, the described change is explicitly connected to the context of the problem.	The solution indicates that the PST described the change in a process or relationship as a function of the relationship between variables in the problem, (i.e., change in the input variable with respect to the change in the output variable); BUT the described change is not explicitly connected to the context of the problem.	The solution does not indicate that the PST considered change in a process or relationship as a function of the relationship between variables in the problem, i.e., change in the input variable with respect to the change in the output variable.
Justifying a Rule	The PST explained why the rule found in the problem works for any number; The justification is explicitly connected to the context of the problem.	The solution indicates that the PST explained why the rule found in the problem works for any number; The justification is not explicitly connected to the context of the problem.	The solution does not indicate that the PST explained why the rule found in the problem works for any number.

Appendix B

Examples of task types

Task 1 (Glencoe/McGraw-Hill, 2005, p. 332)
(*Features elicited or encouraged: #1-7*)
Here is a letter I made in different sizes using small tiles.

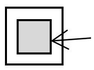
1. Describe how the letter grows from one size to the next.
2. How many tiles would you need to make a letter I of: a) Size 6? b) Size 10? c) Size 38? d) Size 100?
3. Write a rule that helps to predict the number of tiles for any size letter I? You may write a rule either in words or using variables.
4. Suppose you had 39 tiles. What is largest size of I that you could make? Justify how you know



Size 1 Size 2 Size 3

Task 2 (Source unknown)
(*Features elicited or encouraged: #1-5, 7*)
Sally is having a party. The first time the doorbell rings, one guest enters. If on each successive ring a group enters that has 2 more persons than the group that entered on the previous ring how many guests will have arrived after the 20th ring?

Task 3 (Friel, Rachlin, & Doyle, 2001, p. 86)
(*Features elicited or encouraged: #1-7*)
Below is a picture of an in-ground swimming pool surrounded by a border of square tiles.



1. How many 1-foot square tiles will be needed for the border of a square-shaped pool that has edges length *s* feet?
2. In as many ways as you can, express the total number of tiles needed.
3. How do you know that your expressions are equivalent? Provide convincing argument that your expressions are equivalent.

Task 4 (Source unknown)
(*Features elicited or encouraged: #1-7*)
Sherri's monthly bill for phone service is $20. The only additional charges Sherri incurs are when she calls her sister in France, for which she pays an additional $0.15 per minute of phone call. This situation gives rise to a Franco-phone function whose inputs are the numbers of minutes Sherri can talk to her sister and whose outputs are her possible monthly bills.

1. Write a formula for a Franco-phone function.
2. Make a table or graph the Franco-phone function. You should be able to use the table or graph to figure out what Sherri's' monthly bill would be fi she called and talked to her sister for any time between 0 and 30 minutes.
3. What is the rate of change of the Franco-phone function (include units)
4. Explain how the rate of change is shown in your graph.
5. Explain how the rate of change is shown in your formula.

Task 5 (Driscoll & Moyer, 2001)
(*Features elicited or encouraged*: #1-7)
Eight adults and two children need to cross a river. They have a small boat available that can hold one adult or one or two children. Everyone can row the boat.

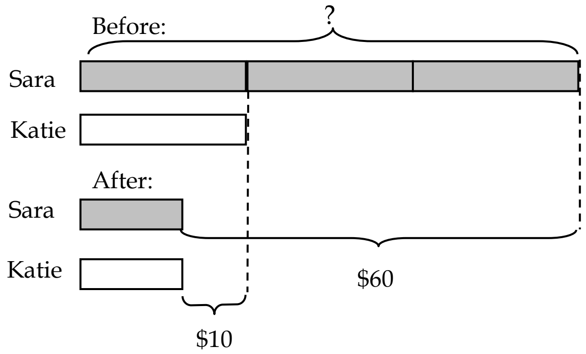
1. How many one way trips does it take for them all to cross the river?
2. What if there were 6 adults and 2 children? 15 adults and 2 children? 3 adults and 2 children?
3. Describe in words how to figure out the answer for this problem if the group of people that crosses the river includes 2 children and any number of adults?
4. How does your rule work out for 100 adults? How do you know?
5. Write the rule for "A" number of adults and 2 children.
6. What happens to the rule you wrote if we change the number of children? For example 8 adults and 3 children? 2 adults and 5 children? Any number of adults and 11 children? Clearly explain.

7. One group of adults and children took 27 trips to cross the river. How many adults and how many children were in the group? Is there more than one solution?

Task 6 (Beckmann, 2008, p. 741)
(*Features elicited or encouraged: #1-7*)
Assume that the following pattern of a square followed by 3 circles and 2 triangles continues to repeat:

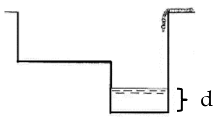
1. What will be the 100-th shape in the pattern? Explain how you can tell.
2. How many circles will there be among first 150 entries in the given sequence? Explain your reasoning.
3. Amanda says that there are 6 circles among the first 10 shapes, and since 150 is 15 sets of 10, there will be 15 x 6 = 90 circles among the first 150 entries. Is Amanda's reasoning correct? Explain why yes or no.

Task 7 (Adapted from Beckmann, 2008)
(*Features elicited or encouraged: #1-5, 7*)
Sara had 3 times as much money as Katie. After Sara spent $60 and Katie spent $10, they each had an equal amount of money left. How much money did Sara have at first?

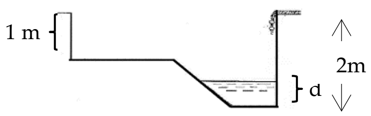


1. Solve the above 6th grade problem with the aid of the diagram and explain your solution.
2. Solve the above problem with an equation. Explain how this solution is related to the method in part one.

Task 8 (Shell Centre for Mathematics Education, 1985, p. 12)
(Features elicited or encouraged: 1, 4, 6)
1. A rectangular swimming pool is being filled using a hosepipe which delivers water at a constant rate. A cross section of the pool is shown below.



Describe fully in words, how the depth (d) of the water in the deep end of the pool varies with time, from the moment that the empty pool begins to fill.
2. A different rectangular pool is being filled in a similar way.



Sketch a graph to show how the depth (d) of water in the deep end of the pool varies with time, from the moment that the empty pool begins to fill. Assume that the pool takes thirty minutes to fill to the brim.

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