Exploring how understandings from abstract algebra can influence the teaching of structure in early algebra

Nicholas H. Wasserman
Teachers College, Columbia University

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Empirical work that connects content knowledge to teaching practice has become increasingly important in discussions around a professional knowledge base for teaching. This paper aims to continue that work, reporting a single exploratory case study of one elementary mathematics teacher. The purpose of the study was to investigate how introducing teachers to advanced mathematics, particularly ideas in abstract algebra, might inform and influence their instructional practices for teaching early algebra topics about arithmetic structure. The findings indicate that instructional practices did change, especially in ways that more explicitly incorporated structure in early algebra with students, and fostered increased reasoning and sense-making about these ideas. Implications for mathematics teacher education are discussed.

Keywords · early algebra · mathematical knowledge for teaching · abstract algebra

Introduction

Teachers must know their subject matter, that much is clear. But what the "scope" of that content knowledge is and how should they "know" it is decidedly less so. Evidence from research (e.g., Monk, 1994) indicates that simplistic conceptions of mathematics teachers' content knowledge (e.g., the number of undergraduate content courses) are insufficient; knowledge for teaching is a more complex construct. Recently, practice-based approaches to teacher knowledge have become prominent (e.g., Hill, Sleep, Lewis, & Ball, 2007; Rowland, Huckstep, & Thwaites, 2005). These approaches maintain that the mathematical knowledge teachers need should be relevant for the professional activities of teaching, such as explaining concepts, sequencing instruction, designing activities, and accessing students' thinking.

Yet defining content knowledge in relation to teaching also raises questions about the scope of mathematical knowledge that may be relevant, especially regarding more advanced mathematics. According to a practice-based approach to teacher knowledge, a school teacher's knowledge of advanced mathematics, such as abstract algebra, would have to translate to their instructional practice in some way. And yet school mathematics teachers should not, in fact, end up teaching their students abstract algebra. This presents a difficult tension to resolve (Zazkis & Leikin, 2010). The paper sets out to explore this issue further, looking at practice-based evidence from the classroom to explore whether a school teacher's study of advanced mathematics plays a role in their instruction, and, if so, in what ways.

Teachers' Mathematical Knowledge

Mathematical knowledge for teaching

In the past decade, various frameworks have been developed to help characterize the ways that mathematical knowledge influences teaching practice. Perhaps one of the more well-known is an expansion of Shulman's (1986) distinction between subject matter and pedagogical content knowledge, Ball, Thames, and Phelps' (2008) Mathematical Knowledge for Teaching (MKT) framework. The MKT framework depicted three subdomains of subject matter knowledge and three subdomains of pedagogical content knowledge; two of these subdomains, horizon content knowledge (HCK) and knowledge of content and curriculum (KCC), were only provisionally included. The provisional subdomain of HCK is most pertinent to this work. Although there have been vignettes depicting HCK (Jakobsen, Thames, & Ribeiro, 2013; Wasserman & Stockton, 2013; Zazkis & Mamolo, 2011), few, if any, empirical studies have been conducted as a means to inform this provisional subdomain, and little consensus about this subdomain has been reached (e.g., Fernandez & Figueiras, 2014; Zazkis & Mamolo, 2011).

The perspective on HCK in this study is related to Wasserman's (2016a; 2016b) conceptualization of local and nonlocal mathematical knowledge. First, I differentiate between the terms "mathematical horizon" and "horizon content knowledge." The mathematical horizon is a relative description of where ideas fall within a mathematical landscape. Namely, the mathematical horizon would consist of those ideas that are relatively farther from the (local) content that a teacher teaches. Because advanced mathematics is outside the (local) scope of mathematics being taught by a school teacher, it is considered part of the mathematical horizon. In this sense, I differentiate between "knowledge of the mathematical horizon" and "horizon content knowledge." The former would be any knowledge that a teacher has about ideas in the mathematical horizon; the latter is more intrinsically linked to classroom practice. Given the focus of this paper, which is about connecting knowledge of advanced mathematics to teachers' classroom practices, this work is related to horizon content knowledge.

In other work, Silverman and Thompson (2008) outlined a two-step cognitive model for the development of mathematical knowledge for teaching: i) the development of *personally powerful* mathematical understandings (which have pedagogical potential); and ii) the transformation of these understandings into mathematical knowledge that is *pedagogically powerful*. Their description of personally powerful mathematics understandings was related to Simon's (2006) notion of *key developmental understandings* (KDUs), which were described as a "conceptual advance… a change in [one's] ability to think about and/or perceive particular mathematical relationships" (p. 362). Adapting this model to consider how teachers' knowledge of advanced mathematics might be influential for their teaching would indicate that the (nonlocal) advanced mathematics first serve as a *personally powerful* understanding of the (local) school mathematics that the teacher teaches. These understandings, then, would have some pedagogical transformation affecting classroom practice. Presuming teachers to have gained some mathematically powerful understandings in light of advanced mathematics, this paper considers more specifically in what pedagogically powerful ways this might be realized.

Knowledge quartet

In contrast to distinguishing knowledge domains, the development of Rowland et al.'s knowledge quartet (KQ) framework (Rowland, Huckstep, & Thwaites, 2005; Rowland, Turner, Thwaites & Huckstep, 2009) stemmed from detailing how teachers' mathematical knowledge was evident in teaching. In terms of this study's analysis, the KQ framework provides an important link: if this study more broadly investigates a particular subdomain of teachers' mathematical knowledge, the knowledge quartet provided a means by which to analyse classroom practice for particular kinds of instructional actions that provide insight into a teacher's use of subject matter knowledge during instruction. The knowledge quartet consists of four dimensions: foundation, transformation, connection, and contingency. Within each dimension, various contributory codes help isolate and describe the particular *instructional* instances that are an indicator of a teacher's content knowledge in that dimension. Instructional instances can be assessed more generally as either 'strong' or 'weak' examples (Rowland, 2012); that is, a particular contributory code indicates an example from classroom practice that speaks either positively or negatively about the teachers' mathematical knowledge in that instance.

Foundation consists broadly of the subject matter knowledge and beliefs acquired in academic study; this dimension differs from the other three in that it represents the foundational mathematics-related knowledge one possesses. I elaborate on four of the seven contributory codes for this dimension that are most pertinent to the report in this study (Appendix A can be referenced for the entire list). Instances coded as *theoretical underpinning of pedagogy* are broadly indicative of some of the foundational beliefs that teachers have both about mathematics and mathematics teaching; a 'strong' example demonstrates drawing on knowledge of mathematics education research or foundational ideas about mathematics in the planning or enactment of a lesson. A strong *awareness of purpose* is an indication from the teacher that they are aware of, and make students aware of, the mathematical purpose of an idea or a lesson. Regarding *mathematical terminology*, the inclusion of correct and precise terminology that encapsulates important mathematical ideas during instruction is also a positive indication. Finally, a teacher's *concentration on procedures*, perhaps without indication of why or with reluctance to alternative approaches, is a poor indication of the teacher's foundational mathematical knowledge. Such actions present in a teacher's practice provide some information about their foundation in subject matter knowledge.

Transformation relates to how teachers make use of and translate their knowledge for pedagogical purposes; two of the four codes in this dimension are discussed. Frequently, teachers engage in some sort of mathematical explanation or *teacher demonstration* for students; examples that are mathematically correct and conceptually appropriate for students are strong indications about teachers' transformational knowledge of mathematics. As for *choice of examples*, the problems and examples that one uses during instruction and as part of homework assignments can be an indication about subject matter knowledge; inclusion of examples that are mathematically thoughtful, intentional, and varied are positive indicators.

Connection regards the teacher's choices about unifying and drawing out coherence between otherwise seemingly discrete parts of mathematics. As a body of knowledge that is notably coherent, mathematics teaching should develop this sense of connectedness; two of the five contributory codes are elaborated upon. Instances when a teacher takes advantage of opportunities to *make connections between various procedures* are one positive indicator of the teacher's mathematical knowledge in terms of connections. *Decisions about sequencing* concerns the coherence displayed across a lesson or sequence of lessons; a positive indication in this regard would be that ideas, strategies, and concepts are ordered progressively to facilitate students' mathematical development.

The last dimension, *contingency*, is representative of how a teacher responds to the inevitable deviations and interruptions that arise in the classroom. One of the four codes is pertinent for this study. Almost inevitably during a lesson, a teacher must *respond to students' ideas*, questions, concerns, etc.; responses that are coherent, well-reasoned, and that directly address and incorporate the student's idea are a positive indication of the contingent nature of the teacher's mathematical knowledge.

Abstract Algebra and Early Algebra

Because this study is concerned with advanced mathematics, particularly abstract algebra, and whether teachers' knowledge of such mathematics is influential on their teaching practice, it is necessary to establish some relationships between the content of abstract algebra and school mathematics. In this paper, the focus is explicitly on connections to early algebra (primarily elementary-school mathematics) and not others, e.g., secondary mathematics.

Abstract algebra is the study and generalization of algebraic structures, predominantly encompassing groups, rings, and fields. Although school mathematics (K-12) does not include explicit study of such ideas, the content of school algebra implicitly draws on these algebraic structures (largely fields and rings). Often, these structural ideas begin developing with early notions of arithmetic. Indeed, scholars emphasize the algebraic nature of arithmetic thinking and conceptualize algebra as much more than techniques to "solve for x." Kaput (2008), for example, delineated algebra as: 1) the study of structures and systems abstracted from computations and relations; 2) the study of functions and joint variation; and 3) the application of modelling languages in and out of mathematics. In terms of these three aspects, this study focuses specifically on the first: the teaching of structure in early algebra.

Structure in early algebra

According to Slavit (1999), understanding structure in early algebra requires attention to some differing aspects of operations, including property aspects, relational aspects, and application aspects. As students progress in school, understanding (or lack of understanding) of structure in numerical contexts frequently carries over to algebraic contexts (Linchevski & Livneh, 1999; Warren, 2003). Yet Morris (1999) also suggested that teachers frequently assume students are familiar with arithmetic structure and give little attention to it in their instruction. Wasserman (2016a) conceptualized four primary areas in school mathematics related to structure in early algebra and their progression across elementary, middle, and secondary mathematics, where teaching may be transformed by teachers' knowledge of abstract algebra (including (i) arithmetic properties, ii) inverses, iii) structure of sets and iv) solving equations). The four content areas elaborated upon in Table 1, which will be referred to as *structure in early algebra* content, were used as the source for determining which lessons from a teacher's practice to observe. Notably, they primarily relate to arithmetic structure, echoing Slavit's (1999) connection to properties, relational aspects, and applications.

Table 1
Description of Structure in Early Algebra content (adapted from Wasserman (2016a))

Content Area	Elaboration in elementary mathematics	Association to abstract algebra
Arithmetic Properties	Properties of addition and multiplication on natural numbers and fractions (positive rational numbers).	Structures in abstract algebra are based on a collection of *arithmetic properties*, which form the foundations for algebraic reasoning.
Inverses	Inverse operations (e.g., addition and subtraction).	The relational aspect of inverse operations is informed by *inverse elements* and the extension of number sets in search of *closure*.
Structure of Sets	Equivalence within number sets (identity), particularly on the set of fractions (positive rational numbers).	The *multiplicative identity*, 1, for example, is important for constructing equivalent fractions, and reducing fractions, an application of structure.

Solving Equations	Number sentences; Numerical reasoning	Assumptions necessary for solving equations, and numerical reasoning about number sentences, are based on the collective importance of *arithmetic properties*, an application of structure.

Methodology

On the whole, this study was primarily interested in the following research question: can an introduction to ideas in abstract algebra influence the teaching of structure in early algebra, and, if so, in what ways? Given the nature of the research questions, an exploratory case study (Yin, 2008) design was used to provide qualitatively rich data from various sources to describe the teaching of structure in early algebra. Notably, the first aspect of the question regards a proof of concept; since very little research has considered empirical data from classroom instruction in exploring the utility of advanced mathematical knowledge for teachers, this aspect is not trivial. This paper reports a single case study analysis in which the selection of the case was intentional; it *supports* the claim that study of abstract algebra ideas can influence the teaching of structure in early algebra. This decision allowed the analysis to focus in more detail on the second aspect of the question: describing how this influence was evident in the classroom.

Briefly, a general overview of the research study is provided. The study incorporated a pre/post design in order to capture instructional *changes* made by the participating teacher. A variety of artefacts were collected to capture a teacher's instruction about *structure in early algebra*, including their planned (submitted lesson materials), enacted (videotaped lessons), and reflected (interviews) practices. The triangulation of data (Denzin, 1978) provided a variety of sources with which to consider particular claims about the participant's instruction. In between the collection of pre- and post-instruction data, teachers participated in a summer course module that introduced them to ideas in abstract algebra, primarily groups and group theory. By design, this helped link potential changes in instructional practice to the study of abstract algebra. In addition, follow-up interviews allowed teachers to elaborate on some of their instructional decisions, which provided a more direct source of data to be able to connect their instruction to ideas learned from the summer module (Figure 1).

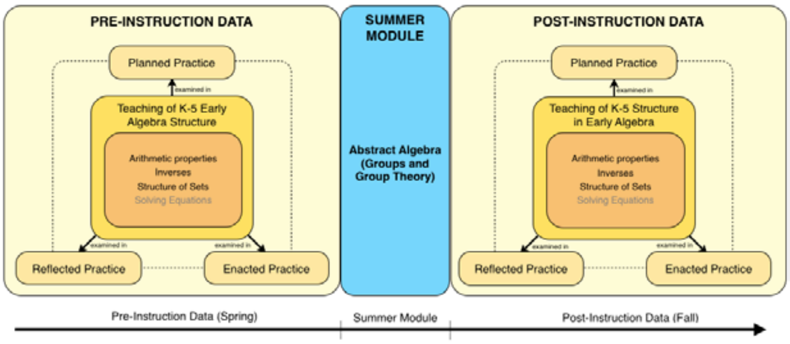

Figure 1. Overall pre/post study design.

Participants

From a larger pool of six in-service elementary, middle, and high school teachers (two from each grade level) who participated in this study, this paper reports on one of the elementary teachers from the original sample. The discussion in this paper is restricted to this one participant for a few reasons: i) the single case was selected because the corpus of data for this participant provided a compelling sense of instructional change from all data sources, a purposeful selection as a proof of concept that instruction in early algebra *can* be influenced by ideas in abstract algebra; ii) the use of a single case analysis affords more detailed analyses of the various data sources, many of which came from classroom teaching episodes, a notoriously complex and rich environment to analyse and describe: the single case study is intended to depict in sufficient detail what may be *possible,* as a means to shed additional light on the difficult question about the relationship between advanced mathematics and teaching school mathematics; and iii) elementary mathematics is even further removed from abstract algebra than middle or secondary mathematics, providing a more provocative connection for the field to consider.

Lewis. During the study, Lewis taught 5th grade at an elementary school in a large urban school district that served a predominantly Hispanic student body (88%) with a large percentage of students eligible for free and reduced lunch (98%). Lewis, a fluent Spanish speaker, had 5 years of teaching experience as an elementary teacher. Having completed a bachelor’s degree in English, he was pursuing both a Master of Arts in bilingual education and a state-level certification as a Master Mathematics Teacher at the time of the study.

Procedures

Pre-instruction data. Data from the participant was first gathered during the spring semester to characterize the teacher’s initial approaches to teaching structure in early algebra. Lewis submitted all lesson materials for three lessons about structure in early algebra (i.e., Table 1), made explicit with specific grade-level standards from the Common Core State Standards in Mathematics (CCSS-M, 2010); the participant was not given an explanation why the selected topics were of particular interest. Table 2 elaborates on the instructional data collected. In addition to the written lesson materials, Lewis videotaped his teaching for two classes. After submission, he participated in a 25-30 minute semi-structured interview (Gibson & Brown, 2009) led by the researcher. The interview protocol probed the rationale for his teaching decisions, including the key ideas informing the lesson design and implementation for each of the three lessons and two videotaped classes.

Summer Module. As part of a mathematics and education summer course taught by the researcher, Lewis participated in a module introducing him to topics in abstract algebra, particularly groups, lasting approximately eight hours over four instructional days. The four group axioms were discussed, including presentations of proofs about some essential properties (e.g., the identity element is unique, cancellation laws). Various examples were introduced, demonstrating a broad application of groups (e.g., groups of symmetries, matrices, integers modulo *n*, polynomials, functions). The module was intentionally mathematical. Much of the content and examples would be similar to an introduction in an abstract algebra course: students looked at small finite group tables, worked with definitions and axioms, considered a variety of sets and operations that were apt to be familiar, but also grappled with more abstract sets and operations. The instructional approach, however, was likely different from a standard abstract algebra course. Although the instructor did lecture at times, students were often engaged in working with others on problem sets intended to further develop ideas about abstract algebraic structures. Implications of groups for school mathematics were discussed in order to draw out and connect algebraic structures to a familiar context, for example, that the steps for solving simple linear equations such as x+5=12 or X*RX=$R2$ highlight the necessity for the operation (e.g., addition) on the set (e.g., integers) to fulfil the four group axioms (closure, associative, identity element, and inverse elements) (see Wasserman, 2014). In sum, the content of the course was focused on introducing abstract algebra concepts, yet the overarching approach to the course also incorporated some principles intended to situate this knowledge in its relation to a broader mathematical landscape. Although participants discussed potential connections to elementary content, there were no explicit approaches for *teaching* school mathematics topics that were included in or advocated for during the module.

Post-instruction data. In the fall semester after the summer module, Lewis again submitted lesson materials and videotaped lessons on content connected to structure in early algebra. Because of the timing of the summer module between spring and fall semesters, the post-videotaped lessons could not be the same as the pre-videotaped lessons. However, the researcher ensured that the content of the lessons was related to the same early algebra categories (see Table 2). This use of different lessons and semesters was intentional, to ensure that the participant really did write and record new lessons, but also had the disadvantage that exact changes to individual lessons cannot be described. Overall, the data provide a good sense of Lewis' approaches to teaching structure in early algebra, since all lessons and materials drew from the same broader content areas. Lewis similarly took part in a post-interview in which he described his teaching decisions using the same interview protocol as before.

Table 2
Description of Lesson Plan Materials and Videotaped lesson data for Lewis

Lewis	
Pre-Lessons	Post-Lessons
Pre1. Multiplication and division as inverse operations (Inverses)	**Post1.** Add and subtract whole numbers (Inverses, arithmetic properties)
Pre2. Addition and subtraction of fractions, with unlike denominators (Arithmetic properties, inverses)*	**Post2.** Strategies for multiplying two 2-digit numbers (Arithmetic properties)*
Pre3. Multiplication of fractions (Arithmetic properties, structure of sets)*	**Post3.** Strategies for and patterns in multiplication (Arithmetic properties)*

*Indicates videotaped lesson

Analysis

In order to answer the research question, the analysis process for each of the lesson materials, videotaped classes, and (transcribed) interviews consisted of two phases. Firstly, the researcher analysed both the lesson materials (lesson plans, worksheets, activities, homework, etc.) and the videotaped classes. This involved parsing each lesson into segments to capture the overarching lesson development (see Appendix B). Then, each of these segments was analysed according to the contributory codes of the four dimensions of the Knowledge Quartet framework. Based on the characterizations in Appendix A, each instance was coded as indicative of either a strength or weakness regarding the teacher's mathematical knowledge in teaching, particularly about structure in early algebra. Broadly, these provide an indication about the four dimensions of the teacher's mathematical knowledge. More specifically, the analysis also affords identification of various instructional changes that occurred in the pre- and post-lessons. Indeed, differences as identified via the coding for the pre- and post-lessons helped lead to the selection of Lewis as a single case study to explore.

A few comments regarding this process are necessary. Firstly, the researcher coded both the lesson plans and the observations. However, because of redundancy across these two data sources, aspects from the lesson plans that were later enacted during the lessons were only counted once. For example, the following excerpt was taken from one of the lesson plans: "Using tiles represent 5 x 7. Have the students find the product. Then ask, what do you think would be different if I write the expression 7 x 5? (Commutative property)." This activity then was observed during the videotaped lesson; this instructional instance was coded once, not twice. Similarly, although in the video the teacher may have said a vocabulary word many times during a short period of time, it was only coded as "use of mathematical terminology" once. Only when the terminology was used in a fundamentally different problem or context was it considered a second instance to code. In essence, the researcher aimed to code instances that were recognizably distinct so as to avoid unnecessary (and potentially unproductive) redundancy. Secondly, given the overlapping nature of the contributory codes, some instructional instances were given multiple codes. For example, the lesson excerpt provided previously was coded as an indication of: i) choice of examples (strong), because the choice reflected a clear *intention,* mathematically (i.e., commutativity of multiplication); ii) choice of representations (strong), because of the representation of multiplication via tiles; iii) use of mathematical terminology (strong), because of the explicit link to commutativity, which encapsulates an important mathematical idea; and iv) theoretical underpinning of pedagogy (strong), because the pedagogy reflected using student ideas and reasoning as a point of departure to discuss mathematics. Thirdly, lessons were submitted because of the potential connection to topics regarding structure in early algebra; however, not all aspects of the lessons were – or should have been – related to this structure. Thus, the researcher's coding was done especially in relation to evincing structural knowledge of early algebra. For example, although "use of mathematical terminology" was noted for any mathematical terminology, the final analysis only considered the use of terminology that could have been related to structure in early algebra, since these aspects of the teacher's instruction were of utmost interest. Lastly, the interview transcripts were used to confirm, refute, add, or delete particular codes based on the participant's accounts for certain events. Modifications occurred, but infrequently. For example, during one instance in the interview the teacher conveyed an "awareness of purpose" about a particular instructional instance that was not evident from the lesson plans or videotaped lessons; the instance was recoded accordingly.

Once coded, the second part of the analysis involved ascertaining whether there was a substantive difference in the pre- and post-instruction. The researcher looked for meaningful differences either in the "quantity" or "quality" (i.e., strong/weak) of particular codes across the corpus of pre- and post-data, probing for the kinds of instructional changes that existed. In particular, for the contributory codes that indicated the most differences between the pre- and post-instruction, the researcher analysed the data using broad themes to characterize the particular instruction conveyed in each contributory code, as well as what it indicated about the teacher's mathematical knowledge for teaching structure in early algebra. The analysis followed a constant comparative coding method (Strauss & Corbin, 1990), in which the researcher engaged in regular reflection to ensure that larger claims about the participant were representative of the evidence leading to them. Once made, both confirming and disconfirming instances were sought to support or challenge particular assertions. The findings and discussion are representative of this iterative process.

Findings

The findings from the pre- and post-data analysis according to the contributory codes of the four dimensions of the knowledge quartet are reported in Table 3. (A break-down of the instructional segments and overall development of each lesson is presented in Appendix B.) Some contributory codes were never used and they have been left out of Table 3. As a reference point, the "counts" are across all three lessons and are intended to convey relatively distinct instructional instances. Also, these only include instances that are specifically related to structure in early algebra; thus, any trends within these lessons may not be reflective of the teacher's instruction more broadly.

Table 3
Analysis of Instructional instances in Lewis' three Pre- and three Post-Lessons

KQ Dimension	Contributory Code	Strong/Weak	Pre-	Post-
Foundation	Theoretical underpinning of pedagogy	Strong (+)	3	8
		Weak (-)	3	1
	Awareness of purpose	Strong (+)		3
		Weak (-)		
	Overt display of subject knowledge	Strong (+)	1	3
		Weak (-)		
	Use of mathematical terminology	Strong (+)	2	7
		Weak (-)	2	
	Concentration on procedures	Strong (+)		3
		Weak (-)	8	2
Transformation	Teacher demonstration	Strong (+)		3
		Weak (-)	4	
	Use of instructional materials	Strong (+)	2	6
		Weak (-)		
	Choice of representations	Strong (+)	2	3
		Weak (-)	2	
	Choice of examples	Strong (+)		3
		Weak (-)	3	1
Connection	Making connections between procedures	Strong (+)	5	4
		Weak (-)		
	Making connections between concepts	Strong (+)	1	3
		Weak (-)		
	Decisions about sequencing	Strong (+)	1	2
		Weak (-)		
Contingency	Responding to students' ideas	Strong (+)	3	3
		Weak (-)	2	1

Based on this initial analysis, it is evident that Lewis' instruction about structure in early algebra changed in some observable ways. For example, with respect to structure in early algebra, his lessons went from including zero strong indicators to several such instances in terms of awareness of purpose, concentration on procedures, teacher demonstration, and choice of examples. Other aspects of his instruction showed relatively large increases in strong instances and fewer weak ones (e.g., use of mathematical terminology). On the whole, most of the meaningful changes appear in relation to Lewis' foundational and transformational dimensions of knowledge, with fewer changes evident in connection and contingency.

Exploring qualitative changes in Lewis' pre- and post-instruction

From this initial large-grain analysis, some of the classroom data is explored further to unpack the "kinds" of instructional changes that were evident. Rather than examples of every category and every code, only the contributory codes where there was a meaningful change between the Pre- and Post-lessons are considered. Explicitly, the following contributory codes are explored: i) none or only weak examples in Pre-lessons, with at least some strong examples in the Post-lessons; or ii) at least four more strong examples in Post-lessons. This analysis attempts to highlight and describe the various facets of the change that were evident in Lewis' instruction.

Theoretical underpinning of pedagogy. Based on his lesson materials and enacted lessons, one aspect of Lewis' pedagogy for teaching mathematics was a strength across both pre- and post-instruction. In all his lessons there were opportunities for students to "do" things, frequently in groups; this was considered a pedagogical strength as it gives students an active (and not just passive) role in the classroom. For example, from his Pre1 lesson, the planned activities included: "1. Students will be given a word problem to solve in a small group" and "4. Students will prepare a short oral description of the procedures involved on the solution of their problem." In his post-instruction, the same kind of pedagogical approach was observed. For example, the Post1 lesson plan included: "Students make pairs and are provided with a small white board and dry erase markers… [and] will take turns to challenge each other in mental addition." Across the corpus of data, Lewis consistently planned for students to actively take part in class.

In addition, however, there was one other facet of his pedagogical approach that changed. His pre-instruction focused on a "teacher-first" model of showing students a desirable approach, and then having students practice similar problems. Indeed, all of his pre-lesson plans explicitly adhered to the following: "I do, you observe; I do, you help; You do, I help; You do, I observe." Although it is a strength that students actively engaged in the lesson, the tasks that students were being asked to do during this time were primarily replicating what the teacher had just showed them. This approach of not taking into account students' thoughts on and strategies for a meaningful problem or task and instead giving them only "practice" problems was considered a weakness in terms of classroom pedagogy in Lewis' pre-instruction. In the post-instruction his approach was different, a "student-first" model. Students were frequently asked to engage in some task *first*, which the teacher *afterwards* helped summarize. For example, his Post2 lesson began with the following: "Two students are working on the following multiplication problem: 37x18. One student arranges the problem like this: 37x18, and the other does this: 18x37. Who do you think is correct? Are the products of these two problems different? Solve and share with your pair." In this example, and in others throughout his post-instruction, there is an emphasis on students engaging in a mathematical task first rather than an "I do, you observe…" approach.

Awareness of purpose. There were no instances in Lewis' pre-instruction for which there was a clear indication of purpose in relation to structure in early algebra. By contrast, there were several instances in Lewis' post-instruction where he made a clear indication of purpose regarding structure in early algebra. For example, in his Post3 lesson his plans included: "Students will explore the multiplication table charts looking for patterns *that could help them to be more effective recalling multiplication facts*" (italics added). This is an explicit indication of a purpose for learning about patterns, which in this lesson were linked explicitly to arithmetic properties. Indeed, during his enactment of this particular lesson, he ended class by concluding that learning the zero-product property and the multiplicative identity property helps students "make predictions [beyond the table] … So, if I multiply 2,000,000 x 0, what is going to be the product?" He also rationalized that learning the commutative property is similarly useful, because "while before you might have to know 144 numbers ["facts"], now you only have to know half." These statements explicitly convey a purpose to students for learning structural ideas about arithmetic; no such instances were evident in his pre-instruction.

Use of mathematical terminology. Lewis' use of mathematical terminology also suggests a meaningful difference between his pre- and post-instruction. There were only two brief instances in his pre-instruction where he used potentially relevant terminology: i) in his Pre1 lesson plan, the content being learned was stated as "multiplication and division as inverse operations"; and ii) during his Pre2 lesson, the following interaction took place during class: "[Lewis] What do you call fractions that represent the same quantity? [Students] Equivalent fractions." Both talking about operations as inverses and fractions as equivalent has some structural sense in mathematics. However, two other instances during his pre-instruction were lacking in use of structural terminology. Both occurred during the video from his Pre3 lesson. The first instance highlighted an opportunity for Lewis to be explicit about the number 1 serving as the multiplicative identity (albeit in the context of division):

Lewis:	Let's pretend that the number 5 has a denominator (writes 5/). If you have to choose a denominator that does not change the number 5, what would it be?
Students:	[Silent]
Lewis:	Like if you have to put a denominator right here, but does not change the number 5, what number would it be?
Students:	[Silent].
Lewis:	What number divided by 5 equals 5? Which one? One. [writes 5/1].

In this instance, Lewis seemed to expect students to be familiar with the idea of an identity element, yet he did not take the opportunity to talk about the important role that 1 plays in multiplication nor did he use explicit terminology to name this idea (not to mention the inaccuracy of his last statement). The second instance was in regards to simplifying fractions, where to simplify 2/20 involved writing it as "(2x1)/(2x10)" and then "we cancel the like numbers." Lewis gave no further explanation as to why these numbers would "cancel" in this situation; notably, the reason factors cancel in fractions is related to a number and its (multiplicative) inverse producing the (multiplicative) identity. Not that instruction has to formally describe every detail but, in fact, his mathematical communication that "like numbers cancel" does not work in other situations, e.g., with additive terms, (2+1)/(2+10). An opportunity to be explicit about an arithmetic property was not realized. One final observation is that this inconsistency with using structural terminology was not true of Lewis' instruction more broadly: Lewis was very mindful of listing specific terminology in his lesson plans ("numerator", "denominator", "LCD", "dividend", etc.) as well as using vocabulary during his instruction. However, his use of terminology related to structure in early algebra was sparse.

In contrast, Lewis used terminology related to structure in early algebra much more frequently in his post-instruction. Nearly all the instances were about specific arithmetic properties. Lewis made clear what particular arithmetic properties were, as well as when particular properties were being applied. He allotted much more time during class to unpacking these ideas with students. For example, from his Post1 lesson:

> Students are given a set of color tiles to model math operations and review some of the basic number properties of addition. Using tiles:
>
> 1. Using tiles represent 5 + 6. Have the students find the total. Then ask, does it make any difference in the result if we switch the numbers to have 6 + 5? Elaborate further. *(Commutative Property);*
>
> 2. Using tiles represent 5 + 6. Have students find the total. Then, using the same tiles, represent 5 + 5 + 1. Ask, does it make any difference in the result if I add 5 + 5 first and then 1? Provide more examples. (*Associative Property)* Elaborate further; and

3. Using tiles represent 9 + 0. Have students find the total. Then ask, what happens to a number, every time you add zero? *(Identity Property).* Elaborate further.

Not only was the terminology of arithmetic properties explicit in his lesson plans, but throughout his lessons he used this language with students. In fact, his Post3 lesson was one long activity in which students explored arithmetic properties as they arose from patterns in a multiplication table. Commutativity, for example, was discussed in terms of symmetry in the table (Figure 2). In contrast to his pre-instruction, the use of terminology related to structure in early algebra permeated his post-lesson data and instruction.

Figure 2. A student exploring the commutative property from the multiplication table.

Concentration on procedures, teacher explanation, and use of instructional materials. The discussion of these three contributory codes has been combined because the instances from which they were coded had considerable overlap. Notably, Lewis' pre-instruction had a large focus on procedures, which is a weakness regarding his knowledge of structure in early algebra. There were a number of instances when he emphasized procedures on his pre-lesson plans, for example, on his Pre2 lesson plan: "Exit ticket. Explain: If you want to add or subtract fractions with unlike denominators, *what would be the first step*?" (italics added). When giving an explanation to students, his emphasis on procedures was evident: "[Lewis] Can we use the LCD here [on multiplication of fraction problem]? [Students] NO! [Lewis] Only when we add and subtract." In this example, Lewis has stated that one *cannot* use the LCD on multiplication of fraction problems, which is in fact incorrect; it simply is not the most efficient way to do so. In trying to help students learn procedures for addition and multiplication, Lewis made over-generalizations, which resulted in incorrect statements. In teaching how to generate equivalent fractions, he used the phrase: "What you do to the top you do to the bottom" (Pre2). This procedure, of course, only works with multiplication, and students were not given an opportunity to make sense of the procedure. Moreover, the notation he wrote on the board (which students later mimicked) was "2x(1/3)…=(2/6)," motioning for the 2 to be multiplied to the top *and* bottom of the fraction, 1/3. Unfortunately, as it stands, the expression is mathematically incorrect. In his videotaped lesson on multiplying fractions (Pre3), the *same* written notation was used; however, in that lesson, the result of "2x(1/3)" was (2/3), *not* (2/6). Students did not question the inconsistency. Lewis' communication and concentration on procedures, which at times were inaccurate, characterize his pre-instruction as providing very few (if any) opportunities for sense-making about structural ideas in early algebra.

In contrast, much less of this kind of procedurally-focused instruction was observed in his post-lessons. In fact, his lesson plans tended to focus on students sharing their own thinking as opposed to following procedures. For example, his Post1 lesson plan stated, "Students should explain their mental process out loud. The focus of the practice is accuracy rather than efficiency. So, *students should be allowed enough time to explain their thinking process*" (italics added). His post-instruction also included different kinds of instruction, not just teacher demonstrations/explanations. This is in part related to a broader pedagogical change, but the difference in his post-instruction is best captured by his aim to go beyond knowing "how" to knowing "why." For example, in his Post2 lesson plan, "After modelling a few examples [of using the grid method for multiplication], the teacher brings up how the *associative and distributive properties make the grid method a feasible approach for multiplication*" (italics added). In this example, Lewis pushed for students not to just be able to "do" the grid method for multiplication, but also to understand why this approach is justified in relation to structural ideas. He also regularly pushed students to go beyond procedures: "[Lewis] This [points at 5x36 written on the board] means that you have five groups of 36. This [points at 5x(30+6)] means something else. What does 5x(30+6) mean to you? [Student] Um, well, we multiply [Lewis, interrupts] I don't want…um, I just want you to tell me what that means to you." In this instance, the student started to provide a procedure of "what to do" with 5x(30+6) – how to use the distributive property – but the teacher interrupted, asking the student to focus not on the procedure but on the underlying meaning of multiplication and the distributive property. This was consistent across his post-lessons.

Choice of examples. Lastly, the final major difference between Lewis' pre- and post-instruction had to do with his choice of examples. Throughout his pre-instruction, his choices of examples were not intentional in terms of building up ideas, particularly structural ones. Mostly, his examples could be described as "drill and kill," that is, he showed one example, and then students were given many other similar examples on which to practice. On a worksheet associated with the Pre2 lesson, for example, the following six problems were given: a) $\frac{4}{3}$ - $\frac{2}{5}$; b) $\frac{14}{5}$ - $\frac{4}{3}$; c) $\frac{3}{2}$ - $\frac{9}{7}$; d) $\frac{7}{4}$ - $\frac{8}{5}$; e) $\frac{7}{10}$ + $\frac{2}{5}$; f) $\frac{8}{3}$ - $\frac{3}{2}$. These are all fine practice problems for adding and subtracting fractions, but none intentionally lead to considering structural aspects that might be explored. During the Pre3 lesson, Lewis discussed examples such as 3x(1/2) and 5x(1/3). Another example was 4x(3/4). Although this example could have been explored in relation to the multiplicative identity, or why some products are whole numbers and others are rational numbers, the example was not used in this way. It was just "another" example.

Throughout his post-lessons, however, there was much more intentionality with respect to arithmetic structure in the examples that Lewis chose. For example, the setting up of the multiplication problem 37x18 in two different ways (in Post2 lesson) was intentionally structured so as to bring out the notion of commutativity. Although it may be obvious to us, for students it is not trivial to explain why 37 groups of 18 objects is the same as 18 groups of 37 objects, or why the standard algorithm results in the same end product but is generated by different sums. Indeed, as a further point, many of the homework exercises Lewis selected were purposeful. For example, in addition to mental arithmetic problems such as "6+12" and "13+5", Lewis included problems such as "34+0", "0+95", and "44+11" and "11+44" as practice. The purpose was clear: Lewis wanted students to develop particular structural ideas about addition. Although there was not necessarily explicit purpose in every problem, on the whole, Lewis appeared to be much more intentional about his choice of examples, particularly as they related to helping students develop ideas about structure in early algebra.

Lewis' attribution of instructional changes

From the corpus of data, Lewis appears to have shifted his instruction methods related to teaching students about structure in early algebra. In particular, five instructional differences were elaborated: his theory of pedagogy changed to adopt a student-first approach; he conveyed a greater sense of purpose for teaching students about structure in early algebra; he used and explored the terminology of arithmetic properties in more detail; he was less focused on procedures and, additionally, explored justifications for them; and his choice of examples more intentionally fostered notions about structure in early algebra. These changes primarily highlight Lewis' additional depth of mathematical knowledge about structure in early algebra within the foundational and transformational dimensions. During the interview, Lewis acknowledged that his pre- and post- lessons were completely different:

Interviewer:	So, would you say that the structure of algebra at all informed you choice of teaching this [Post1 lesson]?
Lewis:	Yes, it changed completely. And with other lessons.
Interviewer:	And how so? Why is that?
Lewis:	Well because in the past I was uncomfortable doing the properties. Actually, I felt that they were just some clinical words that they were not actually useful. Then I realized that you know that it's kind of in the rules for you know basic computation and makes mathematics easier.

By the design of the study, instructional changes were intended to be in relation to the summer module in which Lewis participated. In addition, Lewis himself explicitly attributed this shift in his instruction to the introduction to abstract algebra ideas from the summer module:

> Well for me, the arithmetic properties were just a set of useless statements that students did not need to understand. However, when doing the activity with the [group of triangle symmetries in the summer module], I started to see that the arithmetic properties were more than just unpractical statements. I believe that the fact that we were actually seeing what properties such as closure and inverse looked like [at an advanced level] changed my perception of them.

According to Lewis, exploring the advanced and abstract ideas related to algebraic structures served as a catalyst for changing his instructional approach for teaching early algebra topics.

Discussion

The overarching aim of this study was to consider classroom evidence to investigate whether, and in what ways, a school teacher's study of advanced mathematics can play a role in their instruction. A single case study was used to explore this relationship in greater detail. The corpus of data demonstrates a meaningful change in his instruction for teaching students about structure in early algebra; the qualitative analysis captures five specific differences that existed. Lewis himself attributed this change to the summer module. Still, a critic might be sceptical about linking the specific instructional changes to the abstract algebra ideas in the summer module; perhaps it had nothing to do with learning about groups, and that Lewis simply being involved in any mathematical experience might have prompted these changes.

Instructional changes in relation to abstract algebra

In what follows, I discuss some further observations about the findings, and why I consider it more plausible that the study of advanced mathematics was, in fact, related to the instructional changes that were evident in Lewis' teaching. First, there is Lewis' own attribution: he connected his changed perception about arithmetic properties to activities that are *specific* to abstract algebra. The activity that Lewis referred to in the interview asked students to complete a Cayley table for the group of six triangle symmetries under composition and then solve a simple equation on that group. These tasks asked him to consider not just whether the composition of triangle symmetries maintained specific arithmetic properties but also to justify why that was the case, as well as to identify that solving simple equations on this abstract group are a direct result of the set and operation maintaining these properties. From his perspective, it was the abstractness of the examples, which are unlikely to be studied in other contexts, that changed his own understandings and perceptions about arithmetic properties. Also, Lewis was already familiar with the arithmetic properties of addition and multiplication from his own educational experiences both in elementary school and his preparations to be a teacher. It was not for lack of knowledge about additive and multiplicative properties, then, that his pre-instruction did not include them. It seems unlikely that Lewis studying the additive and multiplicative properties again would have led to the kinds of instructional changes that were evident in this study. By removing arithmetic properties from the familiar territory of "arithmetic" (addition and multiplication), Lewis reported acquiring a deeper understanding, that is, there was something productive about engaging with these structures at an abstract level.

Second, there appears to be alignment with the particular instructional changes that were evident and the content and instruction in the summer module. Two supporting perspectives are provided. First, from the broader KQ dimensions, one of the prominent areas of change was in the foundation dimension, which describes mathematical knowledge that is learned from academic study like a summer module. The fact that many instances were connected to foundational knowledge of structure in early algebra further links these changes to the specific focus and content of the summer module. Indeed, learning abstract algebra is, if anything, most likely to influence one's foundational knowledge about mathematics. Second, the five specific descriptions of changes make sense with regards to the summer module. I describe each of these in relation to either the "content" or the "instructional approach" of the summer module. Three of the ways that Lewis' instruction changed were: i) Lewis communicated an increased *awareness of purpose* for teaching structural ideas in early algebra; ii) Lewis used *mathematical terminology* associated with structural ideas in early algebra much more frequently; and iii) Lewis' *choice of examples*, both in class and in homework assignments, had an increased relation to developing structural ideas in early algebra. Each of these changes is easily linked to the "content" of the summer module. The module's use of solving simple equations to reinforce ideas in group theory also provided an opportunity to consider one "purpose" of arithmetic properties: solving equations. That is, arithmetic properties are not just important for "arithmetic" but are also important for "algebra." Because Lewis came to understand that solving equations in algebra *depends* on properties of arithmetic, there was an additional sense of import given to them at the elementary level at which he taught. His choice of examples reflects on another facet of the module; the use of abstract examples fostered the ability to identify specific instantiations within early algebra content. Lewis, for example, drew out the associative and commutative properties in the context of mental arithmetic, which was not discussed in the module. Grappling to understand abstract ideas (e.g., locating the identity element in a group of symmetries) helped Lewis develop eyes to "see" the structure present in elementary forms. The other two ways in which Lewis adjusted his instruction were: i) a *theoretical underpinning about pedagogy* in using a student-first (rather than teacher-first) model of instruction; and ii) he *concentrated less on procedures*, and focused more readily on why those procedures and ideas were sensible. These changes also can be related to the summer module, but less so to the specific content and more so to the instructional approach used. These two changes were also modelled explicitly in the summer module, where students engaged in activities first, with the instructor providing conclusions afterwards, and consistently went beyond procedures to explore deeper meanings and justifications. By experiencing instruction in mathematics that Lewis himself found valuable, he then used those ideas in his own instruction.

Third, there is additional support from theoretical models about the development of mathematical knowledge for teaching. In particular, Silverman and Thompson (2008) argued that cognitive changes preceded instructional ones. Notably, with respect to advanced mathematics, this would look like the content of advanced mathematics serving as a mathematically powerful understanding (KDU) for the teacher's own local content. Regarding Simon's (2006) description of a KDU, which is "a change in [one's] ability to *think about* and/or *perceive* particular mathematical relationships" (p. 362), the data from Lewis appears to support this kind of mathematical development. In particular, Lewis' ability to *identify* the use of arithmetic properties in mental addition on his own, as well as the reported change in his *perception* of arithmetic properties, both speak to the development of a personally powerful mathematical understanding. Notably, this kind of knowledge is not just related to "knowledge of the mathematical horizon." It was not just any understanding about abstract algebra, but rather specific ones that were related to instructional changes, i.e., "horizon content knowledge." As the data analysis suggests, these cognitive changes for Lewis led to specific kinds of instructional changes that occurred, which depict the ways in which Lewis drew on his increased mathematical knowledge and development to inform his own instruction.

Limitations

Firstly, a single case study is not intended to be generalizable. The particular findings about Lewis' instructional changes are intrinsically linked to the participant, to the particular content covered and instructional approach used in the module, etc. Notably, while using abstract sets and operations is typical to instruction in an abstract algebra course, the specific use of solving simple equations in the summer module was relatively unique to this study. In terms of instructional changes for Lewis, both of these facets seemed to play a role. Indeed, with other participants one might also expect different changes. Lewis, for example, was already attuned to making connections with students (i.e., codes in the connections dimension did not indicate meaningful changes between his pre- and post-instruction); for teachers without such a bent, the various connections made between arithmetic and algebra in the summer module may have prompted more changes in this dimension. A larger, coordinated study with a variety of participants across multiple instructors and institutional contexts would be necessary to provide additional clarity about influences on classroom teaching. Secondly, the case study report draws from a relatively small set of lessons and interviews that capture only a snapshot of the participant's teaching at particular points in time. Additional contextual information on what may have been introduced previously or discussed in more depth in subsequent lessons was, at times, lacking. Further changes in practice could have existed that may not have been captured from the submitted materials; alternately, changes identified in the data gathered could have been atypical, evident in only the submitted lessons. However, the consistent characterization of planned and enacted practices from all of Lewis' pre-lessons and the consistent nature of the qualitative changes identified in his post-lessons supports the idea that other lessons about structure in early algebra would be characterized by similar instruction.

Conclusion

This study contributes to the literature by providing evidence that links a teacher's study of advanced mathematics to instructional changes. Indeed, the relatively far distance between elementary and abstract algebra portrayed in this study is intentionally provocative, broadening the discussion about the relative location of mathematical knowledge that teachers may draw on in their local instructional practice. Although courses such as abstract algebra are a far cry from current elementary teachers' content requirements – and I am not arguing that teachers necessarily should complete such a course – the introduction to some of these more abstract algebraic structures provided a degree of mathematical perspective that shaped Lewis' own understanding of elementary content and, subsequently, influenced his instruction. As such, similar advanced mathematics content may have some place in teacher preparation and/or professional development programs. Perhaps inclusion of an introduction to algebraic structures during instruction devoted to arithmetic properties is appropriate. In sum, I argue that content requirements for teacher preparation and professional development do not necessarily need to be *more*, they need to be *more informed.* The study described in this paper aims to contribute to this growing professional knowledge base by examining links between the advanced topic of abstract algebraic structure and the work of teaching in the context of early algebra. Such empirical evidence linking knowledge to practice is increasingly important as we strive, collectively, to articulate a professional knowledge base for mathematics teachers.

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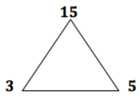
Appendix A: Knowledge Quartet (KQ) coding framework

Below are the four dimensions and contributory codes of the Knowledge Quartet. From extant literature, I have described a "strong" indicator of each code. (Note: i) these are my summaries; ii) "weak" indicators are those that do not fulfil the "strong" descriptions.)

Dimension	Contributory Code / Instructional Instance	Description
Foundation	Theoretical underpinning of pedagogy	Strong: Draws on prominent ideas of the literature and thinking which have resulted from systematic inquiry into the teaching/learning of mathematics; beliefs about 'big ideas' of the discipline of mathematics are aligned with research conclusions
	Awareness of purpose	Strong: Awareness of the mathematical purpose of a lesson and/or what students need to know or practice
	Identifying pupil errors	Strong: Able to identify mathematical errors in students' work and reasoning
	Overt display of subject knowledge	Strong: The mathematics expressed is correct and is related to the topic at hand
	Use of mathematical terminology	Strong: Uses correct terms and precise meanings and symbols; includes mathematical terms which encapsulate important mathematical ideas
	Adherence to textbook	Strong: Flexibly modifies given materials for particular students, differentiating textbook materials for particular populations of students
	Concentration on procedures	Strong: Does not concentrate exclusively on procedures, but also considers why certain procedures are justified; knows multiple ways to do something and is open to alternative approaches
Transformation	Teacher demonstration	Strong: Demonstrations and explanations given to students are mathematically correct and conceptually appropriate for students
	Use of instructional materials	Strong: Materials used as part of instruction are intentional and push students to go beyond knowing how to knowing why
	Choice of representations	Strong: Uses a variety of representations, each of which are mathematically meaningful
	Choice of examples	Strong: Uses a variety of examples that are mathematically thoughtful and intentional
Connection	Making connections between procedures	Strong: Makes productive connections between procedures
	Making connections between concepts	Strong: Makes productive connections between concepts
	Anticipation of complexity	Strong: Anticipates complexity in problems, concepts, procedures, representations, etc.
	Decisions about sequencing	Strong: Intentionally introduces ideas and strategies in an appropriately progressive order
	Recognition of conceptual appropriateness	Strong: Incorporates ideas and methods that are conceptually appropriate; avoids ones that are not
Contingency	Responding to students' ideas	Strong: Coherent, reasoned, and well-informed responses to unanticipated ideas or suggestions from students

	Deviation from lesson agenda	Strong: Spends time probing students in order to find out *why* a pupil comes up with a wrong answer; deeper math knowledge in exploring an unintended mathematical idea; presentation of concept in a new way (unplanned); illuminating with an everyday example
	Teacher insight	Strong: Reflects-in-action and changes tack; explains that something else might work better and why; realizes something important in the moment
	Responding to the (un)availability of tools and resources	Strong: Draws on alternative resources and makes accommodation – when tool fails, finds another way; when tool is there, makes use of it

Appendix B: Segments and development of submitted lessons

Lewis	
Pre-Lessons	Post-Lessons
Pre1. Multiplication and division as inverse operations (Inverses) **Segment 1.** Teacher models a factor triangle, explaining the associated multiplication and division statements.  **Segment 2.** Teacher models the solution of a word problem (*Danny has 132 party favors to divide equally among his party guests. How many party favors will each guest get?*) using the SQUARE problem solving strategy (S=set model; Q=quantify units; U=underline numbers; A=assign groups; R=resolve; E=evaluate), and uses the inverse operation to verify the solution. **Segment 3.** Students work individually on a word problem using the SQUARE strategy (*Cindy has 248 pictures to put in 6 photo albums. If each album page holds 8 pictures, how many pages will she be able to fill?*) **Segment 4.** Students work in groups on 7 other very similar word problems. **Segment 5.** Exit ticket: Explain how can you make sure that the problem you just solved shows a reasonable solution?	Post1. Add and subtract whole numbers (Inverses, arithmetic properties) **Segment 1.** Warmup: Students are given a set of color tiles to model math operations and review some of the basic number properties of addition. Using tiles: • Using tiles represent 5+6. Have the students find the total. Then ask, does it make any difference in the result if we switch the numbers to have 6 + 5? Elaborate further. *(Commutative Property)* • Using tiles represent 5+6. Have students find the total. Then, using the same tiles, represent 5 + 5 + 1. Ask, does it make any difference in the result if I add 5 +5 first and ten 1? Provide more examples. (*Associative Property)*. Elaborate further. • Using tiles represent 9 + 0. Have students find the total. Then ask, what happens to a number, every time you add zero? *(Identity Property). Elaborate further.* **Segment 2.** Teacher models the making 10 strategy using the following two examples: i) 7+8 = 7+3+5 = (7+3)+5 = 10+5 = 15; ii) 25+32 = 20+5+30+2 = (20+30)+(5+2) = 57. **Segment 3.** Students work in groups on 8 mental addition problems – students are instructed to write down the process involved in their solution. Two problems, 44+11 and 11+44, get at commutativity of addition. **Segment 4.** Students work individually on 9 mental addition problems. Two problems, 34+0 and 0+95, get at the identity element of addition.
Pre2. Addition and subtraction of fractions, with unlike denominators (Arithmetic properties, inverses)* **Segment 1.** Teacher models the solution of 1/3 + 1/2 using fraction model method, and then LCD method **Segment 2.** Students work individually to find 3/5 + 1/4 using the LCD method. Students present their work. Teacher then solves the same problem using the fraction model method. **Segment 3.** Students work in groups on 6 other similar fraction problems: a) $\frac{4}{3}$ - $\frac{2}{5}$; b) $\frac{14}{5}$ - $\frac{4}{3}$; c) $\frac{3}{2}$ - $\frac{9}{7}$; d) $\frac{7}{4}$ - $\frac{8}{5}$; e) $\frac{7}{10}$ + $\frac{2}{5}$; f) $\frac{8}{3}$ - $\frac{3}{2}$. **Segment 4.** Exit ticket: Explain if you want to	Post2. Strategies for multiplying two 2-digit numbers (Arithmetic properties)* **Segment 1.** Warmup: Two students are working on the following multiplication problem: 37x18. One student arranges the problem like this: $\begin{array}{r} 37 \\ \times\ 18 \\ \hline \end{array}$, and the other does this: $\begin{array}{r} 18 \\ \times\ 37 \\ \hline \end{array}$. Who do you think is correct? Are the products of these two problems different? Solve and share with your pair. **Segment 2.** Use tiles to represent the following multiplication problems: • Using tiles represent 5 x 7. Have the students find the product. Then ask, what

add or subtract fractions with unlike denominators, what would be the first step?	do you think would be different if I write the expression 7 x 5? *(Commutative Property)* • Using tiles represent 15 x 6. Have students find the product. Then, ask does it make any difference if I break 15 into (10 + 5), and then I multiply 10 and 5 by 6 and add the products? *(Distributive Property)* • Using tiles represent 12 x 5. Have students find the product. Then, ask how would it affect the product if I break 12 into (2 x 6), and then multiply 2 by (6 x 5)? Explain further. (*Associative Property)* **Segment 3.** Teacher models the solution of 3 x 4, 3 x 8, 7 x 4, and 5 x 36 using the grid/area method for multiplication and the distributive property. **Segment 4.** Students work in groups on two multiplication problems: a) 8 x 27; b) 4 x 47. Students present their work using the four products in the distributive property and the grid/area method. **Segment 5.** Teacher models the solution of 26 x 31 using the grid/area method for multiplication and the distributive property. **Segment 6.** Students work individually on other similar problems: a) 21 x 16; b) 21 x 36. Students present their work.
Pre3. Multiplication of fractions (Arithmetic properties, structure of sets)* **Segment 1.** Teacher models the solution of 3 x 1/2 and then 4 x 3/4 using fraction model method **Segment 2.** Students work individually to find 5 x 1/3 using fraction model method. Students present their work. **Segment 3.** Teacher demonstrates algorithm of multiplying straight across on 5 x 1/3, and then 1/2 x 1/3, and then 10 x 1/2. **Segment 4.** Students work individually to find 4 x 3/5. Teacher writes up the fraction model method solution. Students share their answer, 12/5, aloud. Teacher models using TIBO (T=top, I=in, B=bottom, O=out) to find the mixed fraction answer. **Segment 5.** Teacher models fraction model method solution for 1/2 x 1/3. Teacher models the solution of 2/5 x 1/4 using both the standard algorithm and the fraction model method. **Segment 6.** Students work individually to find 3/5 x 1/6 using both methods. Students present their work. **Segment 7.** Students work in groups on 4 other very similar fraction problems (two using the standard algorithm; two using the fraction model	Post3. Strategies for and patterns in multiplication (Arithmetic properties)* **Segment 1.** Warmup: Students consider patterns in: i) PaTtErNs; ii) 1,2,3,__; iii) 5,10,_,20; iv) 2,5,7,10,_,15; v) 9,18,27,_ **Segment 2.** Students are given a multiplication chart of numbers up to 10. Students work in groups to find as many patterns as possible. Groups write down interesting patterns on poster paper, for a class gallery walk. Some of the following patterns are expected to be found: • That the multiplication chart is symmetrical with respect of the main diagonal *(Commutative Property of multiplication).* • That any number times 0 equals 0 *(Zero-product property).* • That any number times 1 equal the same number *(Identity property).* • That the digits in product of number 9 add up to nine (at least the first 10). • That the numbers in the diagonal of the chart are the product of the number times itself. • That the multiples of number 3, 6, and 9 add up to be multiples of 3s. • That the multiples of number 5 end in

method): a) 6 x 2/7; b) 3/4 x 2/5; c) 7 x 5/6; d) 3/4 x 1/5. Students present their work. **Segment 8.** Exit ticket: Explain what is the difference between adding or subtracting fractions and multiplying them?	either 5 or 0. • That the multiples of even numbers end in either 2, 4, 6, 8, or 0. **Segment 3.** Teacher summarizes main patterns, which highlight arithmetic properties of multiplication.

Author

Nicholas Wasserman
Teachers College, Columbia University, 525 West 120th St., New York, NY 10027
nhw2108@tc.columbia.edu