Examining the Impact of a Framework to Support Prospective Secondary Teachers' Transition from 'Doer' to 'Teacher' of Mathematics

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A transition from "doer" to "teacher" for prospective teachers requires them to reorient from thinking about how they do mathematics to engaging with students and their work, understanding student representations, and planning instruction accordingly. To scaffold a transition, we developed a five-step mathematics as teacher heuristic (MATH) model. The study investigated the impact of MATH on the development of teacher candidates' mathematical knowledge for teaching (MKT) and their pedagogical knowledge. Twenty-two preservice secondary mathematics teachers enrolled in a methods course participated in the study. Findings of the study showed that teacher candidates' MKT was engaged as a result of analysis of the student work. While some teacher candidates based subsequent instructional planning work on what they noticed in the student work, others had gaps between what they noticed and their further planning. Teacher candidates' work samples with noticing and noticing with gaps are shared in the results section.

Keywords · preservice teacher education · teacher knowledge · mathematical knowledge for teaching (MKT) · noticing model · mathematics as teacher heuristic (MATH) model

Introduction

One of the overarching aims of a teacher education program is to support "doers" of mathematics in becoming "teachers" of mathematics. This transition requires teacher candidates to reorient from thinking about how they solve mathematics problems to thinking about how others solve mathematics problems. In particular, this means engaging with students and their work, understanding student representations and thinking processes, and posing questions to understand and guide students to move their thinking forward. In order to scaffold this transition, we developed a five-step approach that we call the Mathematics as Teacher Heuristic (MATH): Teacher Candidates (TCs) 1. solve a mathematics task as a "doer"; 2. assess student work samples associated with the same task; 3. construct solution keys for students; 4. develop scaffolded instructional materials addressing students' strengths and challenges; and 5. reflect on the process (Meagher, Edwards, & Özgün-Koca, 2013). MATH incorporates ideas from

Mathematical Knowledge for Teaching (MKT) (Silverman & Thompson, 2008) and the Noticing Framework (Jacobs, Lamb, & Philipp, 2010) in order to provide a set of experiences that require teacher candidates to shift their mathematical view from a 'learner of mathematics' orientation to one embracing 'teacher-oriented' perspectives (Hart, Najee-Ullah, & Schultz, 2004; Meagher, et al., 2013).

There are three important issues in the process of "learning to teach," namely: the influence of the content knowledge, a novice's learning pedagogical content knowledge, and difficulties in acquiring pedagogical reasoning skills (Brown & Borko, 1992). Furthermore, Brown and Borko assert that "one of the most difficult aspects of learning to teach is making the transition from a personal orientation to a discipline to thinking about how to organise and represent the content of that discipline to facilitate student understanding" (p. 221). The first three phases of MATH respond to the need for teacher education program's support of teacher candidates' role change from "doers" of mathematics to "teachers" of mathematics.

Other studies, while acknowledging the difficulty of engendering reflective practices in novice teachers, stress the importance of reflection in teacher education (Chamoso, Caceres, & Azcarate, 2012; Killeavy & Moloney, 2010; Meagher et al., 2013; Schön, 1987; Zeichner, 1996). Teacher candidates' views regarding teaching and learning typically equate learning "with gaining right answers" (Loughran, 2002, p. 41). This naive view of teaching contrasts markedly with teacher comments that emphasise the importance of "opportunities (for students) to be active and think about their learning experiences" (p. 41). Loughran's study illustrates the importance of giving candidates opportunities to face their views then reflect and reconsider them in order to improve their knowledge for teaching. Swafford, Jones, Thornton, Stump, and Miller (1999) echo these findings and recommend the creation of environments for teachers that improve their content and pedagogical knowledge through reflection. The last two phases of MATH support the importance of reflection in teacher education.

The purpose of this study is to help teacher candidates revisit their role as doers (students), cross the border from doer to teacher, and develop pedagogical skills to foster student learning of mathematics through the MATH model with reflection and collaboration. Our study investigates the impact of MATH on the development of teacher candidates' Mathematical Knowledge for Teaching (MKT) and their Pedagogical Knowledge.

Theoretical Framework

Mathematical Knowledge for Teaching (MKT) includes "the mathematical knowledge, skills, habits of mind, and sensibilities that are entailed" (Bass, 2005, p. 429) in teaching mathematics. The notion of MKT was extended by Silverman and Thompson (2008) and Hill, Ball & Schilling (2008), with the former's framework for MKT proposing five steps for developing knowledge that supports teaching of a particular mathematical topic with an emphasis on conceptual understanding. First, a teacher should develop a Key Developmental Understanding (KDU), meaning build their "ability to think about and/or perceive particular mathematical relationships'' (Simon, 2006, p. 362). However, Silverman and Thompson note that developing KDUs of mathematical ideas does not necessarily develop pedagogical understanding. Thus, they propose the second and third steps, "[construction] of models of the variety of ways students may understand the content (decentring) [and having] an image of how someone else might come to think of the mathematical idea in a similar way" (p. 508). This study draws on the construct of MKT and how it relates to teaching of mathematics in developing a "knowledge quartet" of "foundation, transformation, connection, contingency" (p. 259). Rowland, Huckstep, and Thwaites (2005) describe possible stages of teacher candidates shifting their perspective from their own knowledge of mathematics to the concerns and perspectives they must have as teachers of mathematics. In other words, teacher candidates must shift from an understanding of mathematics for themselves to thinking about how someone else understands, engages in, and might be taught mathematics.

Related to Silverman and Thompson's (2008) "decentring" and of particular relevance of the kind of shift we are asking teachers to make is the notion of "unpacking" discussed in the work of Adler and Davis (2006). In one of their examples of unpacking, Adler and Davis present five different student responses to a standard question involving the solving of a quadratic equation and note that, after seeing at a first level of analysis that all the students have found a correct answer, "the teacher will need to unpack the relationship between a mathematical result or answer and the process of its production" (p. 274). Adler and Davis note that the teacher is also faced with the challenge of interpreting the specific strategies used by each student and consider how those strategies, some of which are incomplete or problematic, will be orchestrated in a classroom setting to consolidate the learning of all students. Engaging in such understandings, which are part of the MKT construct, involves the creation of a dissonance whereby teacher candidates are challenged to think about mathematics in ways that are not their own and displace them from the role of "doer" of mathematics.

This "unpacking" is one aspect of "professional noticing of children's mathematical thinking" (Jacobs, et al., 2010, p.169), a powerful analytical lens to foster teacher candidates' knowledge of teaching approaches that emphasise conceptual understanding. Building on the work of van Es and Sherin (2008), Jacobs et al. (2010) provide a framework for analysing the way teacher candidates engage with students and student work. They characterise this "noticing" the three ways: (a) attending to strategies, (b) interpreting understandings and (c) deciding how to respond to understandings. Their work shows that the ability to "notice" in this professional way is not something teacher candidates typically have but can be engendered with sustained professional development. Therefore, they argue, in addition to being an analytic tool, the framework can serve as a tool for self-reflection for teacher candidates and may be useful in supporting teachers' development.

The MATH model guiding this study consists of a five-phase activity designed with a goal of supporting teacher candidates to start their transition from "doer/student" of mathematics to "teacher" of mathematics via tasks using analysis of student work, development of pedagogical approach, and reflection and collaboration. As can be seen in Table 1, five phases of the MATH model are supported by the Noticing Framework and the MKT model.

In the MATH model, teacher candidates complete the five-step process using student work samples collected from authentic classroom settings. The first phase, *constructing an initial solution,* is crucial not only for a mathematics teacher educator to see teacher candidates' orientation and approach to the problem but also for preservice teachers to reflect on their own solution strategies and how that affects subsequent tasks in the process. This phase aligns with "developing KDU" in MKT. The second phase of MATH is *assess/analyse authentic student work samples associated with the same task.* This is the step in which the noticing framework starts (i.e., where the teacher candidates attend to children's thinking). Also, interpreting the mathematical details in student strategies and understandings in the noticing framework is required in the second MATH phase. In order to assess students' work, teachers' own way of understanding mathematics is "decentred," and they must understand a variety of ways in which students may approach and understand the content. In the third phase of MATH, *consider good questions to ask struggling students,* teacher candidates rely on their work from the previous tasks, what they *noticed* in student work and how they interpret the student thinking. Moreover, they start thinking about how to respond on the basis of *student* understandings by forming questions for individual students. This task is associated with two parts of the noticing framework: interpreting the student thinking and deciding how to respond to it. In addition, teachers should be able to imagine activities that might support another person's development of a similar understanding of the math idea and ways to empower students to learn math ideas in the MKT model.

Table 1.
MATH (Mathematics as Teacher Heuristic) Model and Supporting Frameworks

Professional Noticing of Children's Math Thinking (Jacobs et al., 2010)	MATH Model (This Study)	Mathematical Knowledge for Teaching (Silverman & Thompson, 2008)
	Constructing an Initial Solution	Developing KDU
Attending to children's strategies	Assessing Student Work	Constructing models of the variety of ways students may understand the content
Interpreting children's understandings		Imagine how someone else might come to think of the math idea in a similar way
	Considering Good Student Questions	Imagine activities that might support another person's development of a similar understanding of the math idea;
Deciding how to respond based on children's understandings		
	Revising the task	Imagine ways to empower students to learn math ideas
	Reflection	

The fourth phase, *revising the task*, aims at developing scaffolded instructional materials addressing student challenges, difficulties, and misconceptions (gleaned from earlier analyses). This task focuses on the last step of the noticing framework in which teacher candidates decide how to respond to what they *noticed* in an instructional setting; and the last stage of MKT, how to empower students to learn mathematics. The last phase of MATH, *reflect on the process*, is unique to our model. We ask candidates to discuss differences between how they thought about the problem and high school student approaches, thoughtfully considering what they have gained as a teacher of mathematics with respect to each of the 5 basic mathematical practices (problem solving, reasoning, communicating, connecting, representing (NCTM, 2000).

As teacher candidates move through the steps of the process, they must consider a rich task from an increasingly teacher-centric perspective. Assessing authentic student work, noticing misconceptions and nuance in understanding while considering questions to ask struggling students, and constructing possible solution keys are activities that require candidates to interpret a learner's work and consider guidance for a learner (teacher-oriented tasks) rather than solving the problem on their own terms and presenting it for consideration (a learner's or doer's perspective).

Methodology

In order to investigate our research question, the impact of MATH on the development of teacher candidates' Mathematical Knowledge for Teaching (MKT) and their Pedagogical Knowledge, we worked with teacher candidates enrolled in the second semester of a year-long methods sequence designed for prospective secondary mathematics teachers. All 22 teacher candidates enrolled in the course were undergraduates.

The second methods course was designed to build upon candidates' initial experiences with authentic planning and assessment activities from the field experiences associated with the first methods course. In the second course, candidates engaged in a series of problem solving experiences that are designed to reinforce and extend their experiences with secondary school mathematics content while exploring the notion of "rich" mathematical tasks. Specifically, candidates were provided with 6 mathematical problem sets consisting of one or more rich mathematical tasks that are aligned to a particular mathematics content standard from the NCTM CAEP Secondary Addendum document (NCTM, 2012, revised in 2015): Number and Quantity; Algebra; Geometry and Trigonometry; Statistics and Probability; Calculus; and Discrete Math.

The problem sets were explored by teacher candidates in a way that encouraged their growth from "doers/students" of mathematics to "teachers" of mathematics using the MATH heuristic. First, problems were solved in a manner that engaged them as "doers" of mathematics. Candidates began their work on their Algebra Problem Set by solving the Hats and Umbrella Task (Meyer, 1997; see Figure 1).

Data Collection and Analysis

Teacher candidates worked in pairs to solve the Hats and Umbrella Task as they completed the five MATH tasks. Since the student work samples were collected from both pre-algebra and algebra groups, teacher candidates were asked to complete the five tasks specific to these two groups of students. In this study, the pre-algebra group consists of middle school level students who have not taken any formal algebra class, and the algebra group consists of high school students enrolled in an algebra class. The original five MATH tasks are provided below.

Task 1: Constructing a solution: First, you will solve the problem as a student likely would - with your primary goal being to "find a correct answer" (using two different methods). In all cases, you are to support your answers with significant work.

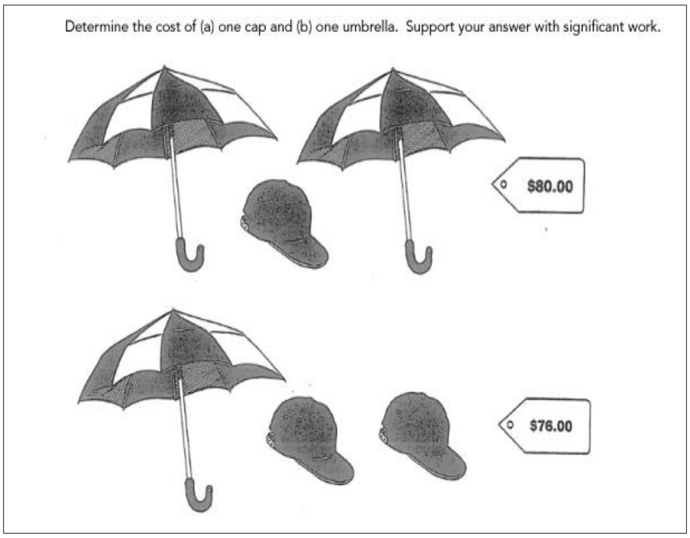

Figure 1. Hats and Umbrella Task (Meyer, 1997)

Task 2: Assessing student work: Next, you will use a pre-fabricated rubric to assess the mathematics understanding/performance of students evidenced in their written work. Moreover, you will identify:

- one successful solution from the Pre-Algebra group;
- one successful (but difficult to follow) solution from the Pre-Algebra group;
- one successful solution from the Algebra group; and
- one positive and one negative trend across the student work samples.

Explain why you chose a student work that exemplifies a successful solution for the Pre-algebra group and so on. This task mirrors more closely the work of a beginning secondary school teacher.

Task 3: Considering good student questions: Prior to revising the task, you will analyse student work carefully, paying particular attention to student stumbling blocks with the task. Brainstorm questions that you might ask a student who is "stuck" while trying to solve the problem. This task also mirrors more closely the work of a beginning secondary school teacher.

Task 4: Revising the task: After assessing student work samples, paying particular attention to student misconceptions associated with the task, you will construct teaching materials (worksheets) that better support students as they explore the problem. You will construct two worksheets that illustrate your understanding of the interconnectedness of mathematical ideas across multiple courses, constructing one worksheet tailored for pre-algebra students and another for first and second year algebra students. The worksheets should be designed to foster student connection making, generalisations, and creativity, while providing students with significant support for solving the tasks you create.

Task 5: Reflection: Lastly, you will consider what you have gained as a teacher of mathematics through your work on this assignment. You will be asked to compare your initial solution strategy with those employed by high school students as well as ways in which your mathematical understanding of the task has grown through the engagement with the 5 basic mathematical practices (problem solving, reasoning, communicating, connecting, representing).

Pairs of teacher candidates completed the five tasks and submitted a completed document to the course instructor. Table 2 summarises data collected from the teacher candidates (TCs) and the data analysis process.

Table 2.
Data Collection and Analysis

MATH Tasks	Data Source	Data Analysis
Task 1: Constructing a Solution	TCs' math solutions	TCs' math approaches to the problem
Task 2: Assessing Student Work	Students' solutions selected by TCs; TCs' analysis of the solutions	TCs' selection criteria for successful/non-successful solutions; TCs' ways of noticing
Task 3: Considering Good Student Questions	Questions posed by TCs	TCs' approaches for future practice to support students' understanding (coherency with noticing)
Task 4: Revising the Task	Tasks revised by TCs	TCs' approaches for future practice to support students' understanding (coherency with noticing)
Task 5: Reflection	TCs' Reflection	TC's interpretation of own noticing and learning

For data analysis, we used theory, investigator, and data triangulation for the trustworthiness of the study (Denzin & Lincoln, 1994). For all five tasks, authors independently coded the data and then met to discuss the codes until they reached 100% agreement.

For analysis of Task 1, the TCs' solutions to the Hats and Umbrellas task were categorised as algebraic (a system of equations and solving either with elimination, substitution, or with matrices), guess and check, graphing, and so on.

For Task 2 the noticing framework was used to analyse the TC's analysis of students' solutions. We coded and categorised the TCs' reasoning for choosing successful solutions and how they identified positive and negative trends in the student work.

For Tasks 3 and 4 we paid particular attention to how candidates built questions for students and revisions of the task based on their analysis of student work. In other words, our analysis focused on whether TCs' choice of questions and their approaches to revising the Hat and Umbrella problem responded to what they noticed or not. Also, Boaler and Humphreys (2005) teacher question types were used to categorise TCs' questions for students. The nine question types were: 1) Gathering information, checking for a method, leading students through a method, 2) Inserting terminology, 3) Exploring mathematical meanings and relationships, 4) Probing, getting students to explain their thinking, 5) Generating discussion, 6) Linking and applying, 7) Extending thinking, 8) Orienting and focusing, and 9) Establishing context. Six codes were used for analysing the teacher candidates' questions: 3), 4), 8), and three components from 1).

Finally, the differentiation framework of content, process, and product (Hall, Strangman, & Meyer, 2003; Tomlinson & Allan, 2000) was used to analyse the revised Hat and Umbrella problem.

- Content refers to what the teacher plans to teach including curriculum topics, concepts, standards, or essential facts and skills.
- Process refers to methods that students employ to make sense and understand the information, ideas, and skills being studied.
- Product refers to the type of student work to be collected, for example reports, tests, brochures, or performances.

In the thematic analysis of data for Tasks 3, 4 and 5, one of the authors first summarised each participant's responses on a spreadsheet. The summary was created using the participants' own words and phrases and included their direct quotes. The authors read the summary spreadsheet looking for patterns and themes across the reflections, then developed themes. Categories of the main themes were taken directly from the language that the participants used in their reflection and formulated by the authors using knowledge of the field. Core categories were central, frequent, and related to other categories with clear implications that allow theory to emerge (Strauss, 1987). Once categories were developed, exemplars of each category were selected for further analysis. These categories are used as subheadings of the Results section, and quotes from reflections are cited in the Results section.

Results

TCs' Math Approaches to the Problem: TCs as Doers

The first task of teacher candidates within the MATH heuristic was solving the mathematics task as doers of mathematics. We asked them to solve the Hats and Umbrellas problem in two different ways to encourage diversity and variety in their solution approaches, particularly since the task is most commonly viewed as a symbolic algebra task for college students. Possible solutions we thought the students might use included various methods for solving systems of equations (e.g., substitution method, elimination method, graphing). Indeed, approximately 81 percent of TC solutions were algebraic solutions centred on a linear system of equations. When asked to provide the second solution, the majority of teacher candidates changed how they solved the linear system of equations (e.g., they used substitution instead of linear combinations). Among algebraic solutions, 23.5% used the elimination method, 47% used the substitution method, and 18% used matrices to solve the linear system of equations. The high percentage of algebraic approaches shows the TCs' current status of mathematical understanding and preferred strategies including privileging of algebra.

Of the non-symbolic manipulation approaches 11.5% used graphing. Three pairs of preservice teachers used the exchanging idea to solve the problem as their one method. This method recognises the price difference between a hat and an umbrella and exchanges a hat or an umbrella in either of the situations to obtain all hats or all umbrellas (see Figure 2). Only one pair of teacher candidates used guess and check as one of their solution paths; but it was their third method in addition to their two other solutions. Many acknowledged that it was difficult for them to come up with the second solution in their reflection write-up and shared that their thinking was focused on an algebraic way:

> When asked to solve the problem at the beginning of the project, we immediately used systems of equations to solve the problem by isolating one variable then substituting it into the second equation to solve for one of the variables and repeating the substitution process to solve for the

> other variable. When asked to solve the hat and umbrella problem in an alternative way, we struggled with how to do so, but ultimately decided to use matrices... However, we had ignorantly ignored the possibility of students using the guess and check method to solve the problem.

Being able to start seeing how someone else might approach a mathematics problem differently was their first noticing act where they started focusing on someone else's mathematics.

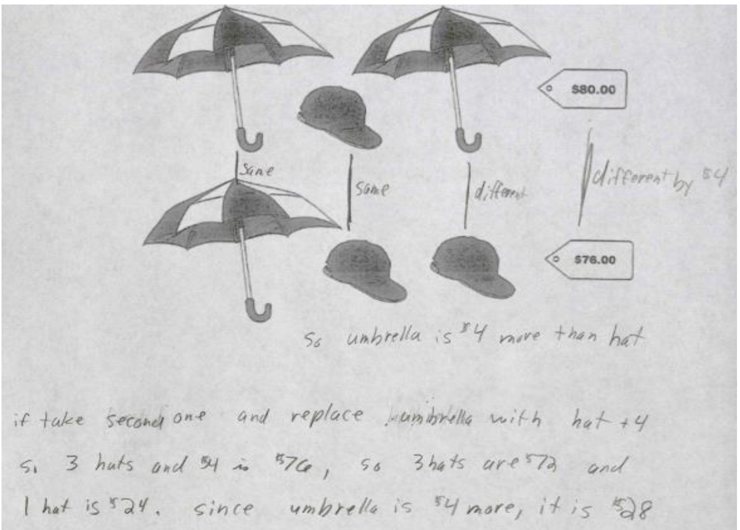

Figure 2. Exchanging

TCs' noticing students' mathematical thinking

Analysing students' solutions consisted of three sub-tasks:

- one successful solution from the Pre-Algebra group and another from the Algebra group
- one successful but difficult to follow solution from the Pre-Algebra group
- one positive and one negative trend across the student work samples.

Teacher candidates had the following criteria for classifying a solution as successful (see Figure 3):

i. if the solution was unique (28% in the Pre-algebra group and 37% in the Algebra group);
ii. if the students used a system of equations (17% in the Pre-algebra group and 21% in the Algebra group);
iii. if the students used a guess and check strategy (17% in the Pre-algebra group and 8% in the Algebra group);
iv. if the students used a graph (17% in the Pre-algebra and 0% in the Algebra group);
v. if the students observed the $4 difference between hat and an umbrella and used the difference to solve the problem (11% percent of Pre-algebra group and 21% of Algebra group).

Teacher candidates did not consider students executing "successful operations" as an exemplar of successful solution for the Algebra group but valued "successful operations" for the Pre-Algebra group. This could be due to the expectation of stronger algebraic thinking at the Algebra level. Teacher candidates identified some student solutions as successful but had struggled assessing them if the solutions were unorganised, showed a lack of plan, misused operation symbols, found the correct answer with unrelated work, or presented multiple solutions.

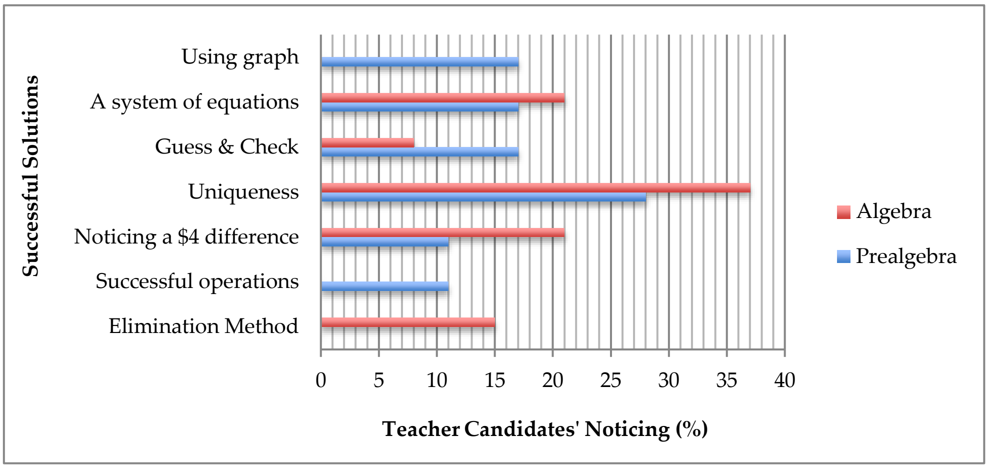

Figure 3. A comparison of successful solutions between PA & A groups

Positive trends that teacher candidates noticed across the high school student solutions were (i) using problem solving strategies (e.g. guess and check), (ii) setting up equations, (iii) being unique (e.g. using graphs (intersection)), (iv) showing effort to solve the problem, (v) explaining their thinking process in words, and (vi) using mathematical language accurately.

On the other hand, negative trends that teacher candidates noticed across the high school student solutions were (i) misunderstanding/misinterpretation of the problem, (ii) using unnecessary or unrelated prior knowledge, (iii) providing incomplete explanation of their thinking process, (iv) decreasing effort as the level of student work progressed, (v) not checking if their answer makes sense, and (vi) presenting their solutions in an unorganised fashion.

In summary, teacher candidates chose exemplars if they see evidence of accurate understanding of the problem, unique ideas in solving the problem, using multiple problem-solving strategies, logical thinking process and organised solution presentation, and checking if ideas make sense mathematically. This stage of MATH provided not only an opportunity for TCs to start decentring from their own thinking as a doer but also provided a foundation on which to base TCs' further instructional decisions, from choosing questions to uncover a student's thinking to revising the task.

Entering into a Teacher's Mind

In the third section of the MATH model teacher candidates were asked to pose good questions for struggling students. Specifically, the questions should be built upon what they *noticed* in student work and their interpretations of student thinking, forming questions to support the unique learning needs of individual students. Both questions posed by TCs and their explanation for the questions are shared in this section.

Some questions are not higher order thinking questions but the TCs' rationale for asking the questions was to respond to the needs of the students. For example, one student left their paper blank, and teacher candidates ask a broad and general question, "There are several problem-solving processes that can be used to solve this problem. Explain how you might begin to process the problem." This pair of teacher candidates explained their rationale for asking the question was for the students to "at least have an idea of what types of processes could be used to solve the problem, as well as why, even if the student does not show any work or work towards an answer."

Table 3 summarises the frequencies of question types that teacher candidates chose to ask students to help them think about next steps based on the work they submitted for the problem. Since the assignment did not suggest or limit the number of questions that teacher candidates could ask, the number of questions written from each pair of TCs differed from 7 to 20 questions.

Table 3.
Question Types

	Question Types	Description	Frequency
General and leading questions	Gathering information	Require immediate answer Want direct answer, usually right or wrong	6
	Checking the work	Rehearse known facts or procedures	18
	Leading students through a method	Enable students to state facts or procedures	27
Potentially Scaffolding questions	Exploring math relationships/ meanings	Points to underlying mathematical relationships and meanings Makes links between mathematical ideas and representations	9
	Probing, getting students to explain their thinking	Clarifies student thinking Enables students to elaborate their thinking for their own benefit and for the class	39
	Orienting and focusing	Helps students to focus on key elements or aspects of the situation in order to enable problem solving	40
	Total		139

General and leading questions. Approximately 4% of questions required students to read the problem again and gather given information. For example, a pair of teacher candidates posed a question "What relationship do you see when looking at this picture?" in order for students to "pick up on patterns or relationships that might help start their planning. There was no work shown, but just an answer." Thirteen percent of the questions were checking for an answer questions. One pair of teacher candidates asked students: "If each hat costs $25.33 and each umbrella costs $26.66, how much will 2 umbrellas and one hat cost? How can we check to make sure that your answer is correct?" These questions have a clear focus on checking the answer. Teacher candidates stated that these questions "can help the student check their work and see that their answer is either correct or incorrect." Nineteen percent of the questions were leading-type questions which provided more specific guidance for the students. Teacher candidates stated the key elements of the problem for the students instead of helping students to see the key elements of the problem by themselves. For example, teacher candidates asked students "Since the umbrella is $4 more than the hat, how is this useful in finding the individual price of each item?" They reasoned that "this question should trigger the student to check their final answer to make sure the two prices have a difference of $4." Another example of guiding and leading students towards a method question was "Is there any way we can set up an equation for each row so that we can compare equations?" These teacher candidates explained the reason for asking this question is that "this should get the student to separate the problem in half and set up two different equations. Then the student should know to solve the system of equations."

Some questions were directly suggesting what students should do: "Would it make more sense to write the problem in equation form?" or "Can you use the two equations that you created to solve this problem?" Teacher candidates explained that "this could help the student go back to some of the work that they showed but did not use to solve the problem. This could get them thinking about the importance of the equations in this problem and what they can do with the two correct equations that they created and how they could help them come to an answer. Maybe if s/he knows that the two equations that they wrote down were correct and useful, they would see how they could use them to find the answer." Some questions posed by TCs were very general or vague without paying close attention to the actual student work; "There is a cost indicated on the two price tags. What do these two prices represent?" "How would you approach the problem by guess and check?" "How would you solve this problem?" or "How else could you solve this problem?" Teacher candidates argued that "by asking this question, s/he could look back at their work and try to solve the problem in a completely new way and find that the hats and umbrellas cost more than s/he had originally thought. If they solve a different way, they could see that their first answer does not make sense and why it does not make sense."

Potentially scaffolding questions. Even though questions in the second three categories are higher level questions than the first three categories, teacher candidates were generally not able to pose scaffolding questions. However, the questions in the second three categories showed the potential that teacher candidates could respond to students' needs with scaffolding questions.

The majority of the questions posed by TCs were 'orienting and focusing' and 'probing, getting students to explain their thinking' questions (56% together). A typical example of orienting and focusing question was:

> Can the price of the hat (or umbrella) change from the first price tag to the second or is this price supposed to remain the same?

Teacher candidates explained the reason for asking this question was that "the student sets the hat to cost $20 for the first price tag and $23 for the second price tag. It is clear that the student did not understand the premise of the problem."

Some teacher candidates had a clear of focus on student thinking, and majority of their questions were 'probing, getting students to explain their thinking' questions:

> Question: What was your reasoning for doing (76+80)/3? (TC's Explanation: This work was at the start of the page and is assumed to be the beginning of the work.)
>
> Question: What does the information in your table represent? (TC's Explanation: Student had arrow signs and dollar signs for hat and umbrella but gave no explanation to this.)
>
> Question: How did you arrive at your final answer? (TC's Explanation: The final answer does not match with the work shown and the final answer does not make sense with the given information.)

These questions follow up on student thinking without any guidance and leading. For instance, even though the aim of the last question is help students see that they are incorrect, this pair of teacher candidates did not ask a "checking for an answer" question as some of the teacher candidates did.

Since the aim of this part of the MATH heuristic was helping candidates pay closer attention to student thinking and consider how to move the student's thinking forward, resisting the tendency to simply coach the students in the TC's preferred method, it is understandable that teacher candidates pursue students with incorrect or incomplete answers. However, how they fostered student thinking with their questions differed. Some teacher candidates chose to provide less guidance. With explain your thinking or orienting and focusing questions, they tried to unpack student thinking further with the aim of building on student thinking. On the other hand, other teacher candidates made students face their incorrect answers with their checking answer questions and provided more guidance in the form of overly leading questions.

TCs' Approaches for Future Practice: Differentiating Mathematical Tasks

After assessing student work and having a better understanding of student strategies through the analytic analysis, teacher candidates constructed two separate adaptations of the original "Hats and Umbrellas" worksheet - one fostering student exploration at Pre-Algebra level and the other at Algebra level. The task was that each worksheet should foster student connection-making through multiple representations (e.g., worksheets foster symbolic and graphical work) and provide students with significant support for solving the task (without being overly leading).

Interestingly, teacher candidates did not modify the mathematical content part of the original Hat and Umbrella problem but added leading questions demonstrated in the previous section. While promoting the guess and check strategy in the pre-algebra groups, TCs guided students to use algebraic approach in the algebra group - i.e., starting with the guidance on how to setup the equations (See Figures 5 and 6 for examples).

e. Now use the tables below to organize your guess and check process; using the same as we used in our example tables.

Table 1:

Hat Price	Money left over after the cost of the hat	Umbrella Price
20	60	30

Table 2:

Umbrella price	Money left over after the cost of the umbrella	Hat price
30	46	23

Figure 5a. Revised Worksheet Example

a. Write the equations from the given information. (Let u= price of umbrella and h=price of hat.)

b. What is the price of one hat?

c. What is the price of one umbrella?

d. How do you know this is the correct solution?

Figure 5b. Revised Worksheet Example 1

A common pattern in *process* differentiation is reflected in the idea that a "guess and check" approach is appropriate for the Pre-Algebra group and a system of equations is for the Algebra group. These perceptions might have been based on teacher candidates' findings from the analytic attention to the student solutions: Guess & check was the main problem-solving strategy Pre-Algebra students used, and a system of equations was mainly used by the Algebra students. We see that candidates encouraged students to employ algebraic solutions (i.e., privileging algebra). This also affected the revision of the worksheets for those different groups of students (see Figure 5). In the revised worksheet for the Pre-Algebra students, TCs pursued the guess and check method with pre-set tables for students to fill (see Figure 5a). The worksheets for the Algebra students were centred on the use of system of equations (see Figure 5b). This way of differentiating the process shows that teacher candidates were inclined towards reducing the cognitive demand of the task as a method of differentiation.

Figure 6 shows another differentiation in *process*. Again, the biggest difference is that Questions for the Algebra group focused on solving the problem algebraically whereas questions for Pre-Algebra group emphasised making sense of their solutions by asking "What happens…" As seen in Figure 6, teacher candidates paid more attention to clarifying the given information or providing additional hints for Pre-Algebra students, and leading Algebra students to solve the problem algebraically. Both Pre-Algebra and Algebra questions required students to explain their thinking/solution by asking why or why not; exploring math relationships/meanings questions by reminding students to prove their answer is correct; and clarifying the given information from the beginning.

Revised Worksheet for Pre-algebra	Revised Worksheet for Algebra
Determine the cost of (a) one hat and (b) one umbrella. Support your answer with significant work.	Determine the cost of (a) one hat and (b) one umbrella. Support your answer with significant work.
	
1. Why does the first set of objects cost more money than the second set of objects? Specifically, which object costs more? 2. What happens to the price-tag of the first set of objects if the hat is replaced with an umbrella? 3. How does question 2 help us determine the individual prices of the objects? 4. What are the prices of the two objects?	1. Build a system of equations labelling x as an umbrella and y as a hat. 2. Use the Elimination method or the Substitution method to solve the system of equations. Support your solution by graphing both equations. 3. What are the prices of the two objects?

Figure 6. Revised Worksheet Example 2

With regard to differentiating the *product*, teacher candidates considered that controlling the total number of solutions (*product* differentiation*)* was a good way to differentiate instruction. Most teacher candidates believed that using multiple solutions requires a higher level of mathematical knowledge and skills and that, therefore, asking; one solution for the Pre-Algebra group and two solutions for the Algebra group.

In summary, our analysis of the revised mathematics worksheets for two different groups of students was based on a commonly used differentiated instruction approach, the framework of *content*, *process*, and *product* (Hall et al., 2003). Teacher candidates did not make any changes in *content* but revised the *process* and the *product* of the mathematics worksheet to differentiate tasks for Pre-algebra and Algebra groups. Their revisions, as can be seen above, were less than convincing and, therefore, teacher candidates’ understanding about differentiation suggests teacher educators need to pay more attention to this topic.

TCs’ Approaches for Future Practice: Supporting Students’ Understanding

During the analysis of the transformation from “doer” to “teacher,” using the noticing framework (Jacobs et al., 2010) and the MKT model (Silverman & Tompkin, 2008), two major themes in the process of transforming were revealed: (a) teacher candidates building on what they noticed in the student work in the remaining parts of the MATH heuristic consistently; and (b) teacher candidates not building on, or having gaps in what they noticed, that then impacted on their work in the rest of the MATH heuristic.

Building on noticing as coherent theme. Two main themes surfaced for the majority of teacher candidates' analyses of student work: (a) students' justification of (checking or not justifying) their approaches/answers and (b) students' use of problem solving strategies. While assessing the students' solutions, teacher candidates considered logically justified solutions successful and those that lacked justification as unsuccessful, as the following comments from two TCs suggest.

> A negative trend I saw was a lack of asking, "Does this make sense?" I feel after one completes a math problem, this question is essential to ask of the final answer. If the students asked this question after they completed the problem, I think many would have reconsidered their work. [And then in the next section they used the following question:] How did you arrive at your final answer?
>
> It seems as though they did not understand that there is a price difference between the umbrellas and hats. Simply checking their work (for example, adding the price of two hats and an umbrella and seeing if the total is $76) could have cleared this problem. [And then in the next section they used the following question:] How can we check to make sure that your answer is correct? [Moreover, in the revised version of the worksheet, they asked] Does the cost of one umbrella and one hat that you found make sense with Jonathan's purchases? With Mark's purchases? Show why your answer satisfies the problem.

Teacher candidates also noticed how students used problem solving strategies such as the guess-and-check approach or a system of equations. Guess-and-check was the main problem-solving strategy Pre-Algebra students used, and a system of equations was mainly used by the Algebra students. Consider, for instance, the following quote taken from the written work of one pair of TCs:

> There are two prominent solution trends that occur throughout the student work. They are "Guess and check" and "systems of equations." The students in level 1 tend to lean towards "guess and check," whereas, the students in level 2 tend to lean towards solving systems of equations. … Both of these methods are excellent solution strategies to this problem. Guess and check is based on the exhaustion problem solving method. It takes a long time to accomplish, but it should eventually work. Systems of equations help students see the relationship between the hats and umbrellas.

We interpret the TCs' mention of "exhaustion" in the previous passage as negative, since such an approach is construed by them as time consuming. Note that the TCs fail to mention whether students used "blind" guessing (i.e., without a pattern) or using informed guessing or "guess and improve." One pair of teacher candidates used what they noticed in their revision of the task for the Algebra group and encouraged students to use equations:

1. Build a system of equations labelling x as an umbrella and y as a hat. [They should have labelled x as the price of an umbrella not an umbrella]
2. Use the Elimination method or the Substitution method to solve the system of equations.

Another pair also noticed that the guess-and-check approach was mainly used by the Pre-Algebra group and a system of equations was used by the Algebra group. One set of questions that they created for an Algebra student included the following:

1. How did you come up with the equation and how does it help solve the question?
2. Can you come up with any more equations to relate the given information?
3. How can you use these equations to solve the problem?

Noticing with gaps. As mentioned in the previous section, some teacher candidates' practice ideas were based on what they noticed in students' solutions. In other words, they posed questions to students and revised tasks based on what they noticed from students' solutions and their interpretations on how students understand concepts initially. However, some teacher candidates could not decentre their own understanding to students' and eventually to practice. Decentring is "the uniquely human ability to differentiate one's own point of view from the point of view of another" (Wolvin & Coakley, 1993, p. 178) and can be critical for teachers (Adler & Davis, 2006).

An example of a pair of teacher candidates whose noticing did not transform to practice was evidenced in two different phases. This pair of teacher candidates valued uniqueness of a students' solutions but asked overly leading questions and provided overly leading guidance in practice. In their analysis of students' solutions, these teacher candidates commended students' ways of solving the problem, "There was definitely a different strategy used than most students." But then constructed overly leading questions and used step-by-step guidance to lead the students to a preferred solution.

> Determine the cost of one cap and one umbrella. Start by splitting up this problem into two separate parts.
>
> a. Focus on just the top row of items. How could you create an equation using the images and price from the top row?
>
> b. Now focus on the bottom row of items. How could you create a second equation using the images and price from the bottom row?
>
> c. Look at one of your equations. Solve for your variable that represents the hat.
>
> d. Use your solution from part c. to plug into the second equation.
>
> e. Solve for the variable that represents the umbrella.

Another unsuccessful example of transformation was teacher candidates valuing problem solving strategies but devaluing and discouraging students using those problem-solving strategies (e.g., guess and check) in practice. A pair of teacher candidates noticed that students were using a variety of problem solving strategies and valued the approaches over using algebraic equations:

> Most of the students in Level 2 (Algebra class) used either a system of equations or guess in check. I think that Level One (Pre-Algebra class) students had more creative ways of solving this problem.... level two students had already learned how to solve math problems using variables and when they look at the problem they think to use variables, and if they cannot think of the equation or how to solve it, students usually resort to the guess & check method which can be just as effective.

But then they discourage students for using the guess and check approach explicitly in their practice:

> Given the two rows of total prices, determine the cost of one hat and one umbrella. The hats and umbrellas are from the same store! Therefore, all the umbrellas cost the same, and all of the hats cost the same. (A hat cost differently than an umbrella).
>
> Include a different method other than "guess and check" to find the solution. Explain your thinking process systematically and also include a written explanation.
>
> Double check your answer for verification. Write neatly and be clear in your explanation.

Teacher candidates who lacked the capacity to decentre did not transform their own Key Developmental Understandings of the concept to an understanding of how this Key Developmental Understanding could empower their students' learning of related ideas. Consequently, instead of developing practice coherent with their noticing, these teacher candidates explicitly imposed their preferred solution methods on students rather than supporting students to develop their own thinking and their own solutions. Possible causes of unsuccessful transformation include (a) weak KDU, (b) unsuccessful noticing, and (c) inadequate decentring.

Discussion and Implication

Despite many years of the "reform movement," broadly understood, being an orthodoxy in schools of education, a great many teacher candidates come to methods classes in mathematics education with traditional ideas about transmission of content, privileging of algebra etc. In our work we are concerned with disrupting this mindset and helping teacher candidates make the transition from "doer" to "teacher" of mathematics. As is evident from the data analysis above this is a far from easy task. However, structures can be put in place that facilitate this transition.

The evidence of this project shows that detailed consideration of authentic high school student work on a problem familiar to the teacher candidates, is not a natural mode of mathematical activity for teacher candidates. Therefore, a clear implication of this study is that the use of such work is important as a transitional tool. Teacher candidates need explicit practice in considering mathematics from another's viewpoint and the obvious limitations in their ability to do so, as evidenced in the analysis above, only underscores the need for this kind of work.

A further implication of this study is the importance of engaging teacher candidates in stages beyond analysing authentic high school student work and into considering next steps. It is evident from the analysis that teacher candidates' tendency is often to reframe the task on their own terms, e.g. privileging algebra, rather than seeking to develop the thinking exhibited by the high school students. It is important that this tendency is laid bare so that it can be interrogated and disrupted.

Teacher candidates' limitations in considering other viewpoints, and developing those viewpoints are compounded and exhibited further in their development of mathematical tasks. Most prominently we see an unsophisticated and underdetermined conceptualisation of differentiation with their primary view of differentiated instruction being one that depends on the number of "clues" given to students rather than an approach where tasks are inherently differentiated. The implication here is that teacher candidates need opportunities to give consideration to how tasks with inherent possibilities for differentiated learning can be constructed.

Perhaps the most important implication of the results of the project is the extent to which many candidates did show recognition of the important issues highlighted in the MATH heuristic in the reflection portions of the assignment. The MATH model includes reflection on the overall process of the activity including reflecting on preservice teachers' reflection on key developmental understandings of their own and their students and their plan of practice. We found the Noticing Framework (Jacobs, Lamb, & Philipp, 2010) to be a useful analytic tool in the study and it may well be useful to share the Framework with teacher candidates as part of the reflection process. For practicing teachers, reflection could be expanded to reflecting on students' learning and practice in the context of instruction (Clarke & Hollingsworth, 2002; Steffe, 1994). Through reflecting on their understanding and practice, Piaget's reflective abstraction, pedagogical action can be transformed into a pedagogical understanding. Some propose that reflective abstraction should be part of the goal of teaching (Simon, Tzur, Heinz, & Kinzel, 2004). Reflective abstraction can be "incredibly useful as a guiding heuristic in a search for insight into mathematical learning" (Steffe, 1991, p. 43), and "seeking ways to facilitate reflexive abstraction is the key to fostering growth" (Gallagher & Reid, 1981, p. 175). Thus, we would argue that the reflection portions of the MATH heuristic are in many ways the key aspect that allows for a trajectory of development as the teacher candidates engage with the heuristic.

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