Extending the notion of Specialized Content Knowledge: Proposing constructs for SCK

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While it is widely believed that *Specialized Content Knowledge* (SCK) is essential to effective and quality mathematics teaching, the specific constructs that compose SCK remain underspecified. This paper describes the development and use of a new framework that extends the notion of SCK. The framework was trialled with a cohort of 90 first year Bachelor of Education (Primary) pre-service teachers who enrolled in a regional Australian university. The pre-service teachers undertook a mathematics test, which required them to address school students' misconceptions and to explain specific mathematical concepts. Resultant data (i.e., the pre-service teachers' responses to the written test) provided an empirical basis for the proposed constructs of SCK. The analysis of the data allowed insight into the central question: whether the proposed framework enables researchers to identify the constructs of SCK in the pre-service teachers' responses to a written test which examines their SCK. Ultimately, we aim to conceptualise the constructs of SCK through elaborating the theoretical and empirical basis.

Keywords evaluation of mathematics content knowledge for teaching • mathematical explanations • mathematical representations • mathematics teacher education

Introduction

Teachers' mathematics content knowledge and pedagogical content knowledge are crucial to their capacity for providing effective teaching (Ball, Hill, & Bass, 2005; Ball, Thames, & Phelps, 2008; Cai, Mok, Reddy, & Stacey, 2016; Hattie & Donoghue, 2016; Krainer, Hsieh, Peck, & Tatto, 2015; Silverman & Thompson, 2008). Research of pre-service teachers' general mathematics content knowledge and mathematics content knowledge for teaching in particular has significantly increased over the last decades (e.g., Baumert et al., 2010; Betts, 2011; Krainer et al., 2015).

Although mathematics educators have been paying special attention to designing relevant and practical pedagogy subjects for pre-service teachers, the nature of mathematical knowledge required by teachers, and the measures of pre-service teachers' content knowledge for teaching, are still unclear (Baumert et al., 2010). For example, common questions asked by pre-service teachers include: "What mathematics content knowledge should we learn for teaching primary school students? What knowledge should be used when we explain a mathematics concept to a child? How can pictures or manipulatives be used effectively to explain a concept?" (S. Brodie 2016, pers. comm., 23 August). In summary, it is challenging for teacher education programmes to determine the most appropriate curriculum for mathematics content knowledge. In addition, tutors of mathematics pedagogy subjects reflect that they do not know "what conceptual understanding they should look for in pre-service teachers' responses to mathematical tasks" and it is difficult to determine what kind of explanations and representations are outstanding, partially acceptable or unacceptable (D. Howarth 2013, pers. comm., 20 May). Ball et al. (2008) have named this particular type of knowledge *Specialized Content Knowledge* for mathematics teaching.

In Shulman's work (1986), he suggested that teacher knowledge consisted of subject knowledge, pedagogical content knowledge and curricular knowledge. Ball et al. (2008) extended Shulman's notion of *Pedagogical Content Knowledge* (PCK) for teaching mathematics and suggested that PCK is a set of knowledge that facilitates teachers to select mathematical tasks, representations and explanations, to interpret classroom discourses and to analyse students' errors and difficulties. While PCK has been proven to have predictive power for student progress and quality of instruction (Baumert et al., 2010), the researchers however concluded that "PCK is inconceivable without a substantial level of CK [subject knowledge]" (p. 163) because subject knowledge is essential to PCK (e.g., Cai et al., 2016; Krainer et al., 2015). In particular, there has been growing interest in, and a marked increase in the attention given to, one of the components of subject knowledge, *Specialized Content Knowledge*. Ball et al. (2008) offered a definition of *Specialized Content Knowledge* for mathematics teaching. They defined *Specialised Content Knowledge* (SCK) for mathematics teaching as "the mathematical knowledge not typically needed for any purpose other than teaching" (Ball et.al., 2008, p. 400). SCK is different from *Common Content Knowledge* (CCK) which is referred to as the mathematical "knowledge of a kind used in a wide variety of settings – in other words not unique to teaching; these are not specialized understandings but are questions that typically would be answerable by others who know mathematics" (Ball et al., 2008, p. 399). In brief, SCK is a particular type of mathematical knowledge and skills unique to teaching, such as conceptual and procedural understanding of mathematics and recognising students' patterns of errors (Ball et al., 2008; Ball, Thames, Bass, Sleep, Lewis, & Phelps, 2009). Hill, Rowan and Ball (2005) further elaborated the meaning of this type of knowledge needed by teachers by stating that it is an understanding and a skill that requires teachers to provide explanations, analyse student responses, and use appropriate images to represent concepts. In summary, SCK refers to teachers' knowledge that requires understanding of mathematics different from the mathematical knowledge needed by other practitioners of mathematics (Silverman & Thompson, 2008). For instance, an accountant and a doctor do not need to give a mathematical reason for finding a common denominator when adding fractions but this reasoning is a teacher's natural work in classroom teaching.

Despite Ball et al. (2008) having offered a definition of *Specialized Content Knowledge* (SCK) and indicating a need for refinement and revision, few researchers (such as Ball et al., 2008; Ball et al., 2009; Baumert et al., 2010) have developed some descriptions of what teachers should know in this domain. While it is widely believed that SCK is essential to effective and quality teaching, the specific constructs that compose SCK remain underspecified. More importantly, even when assessments of pre-service teachers' SCK are authentic and thoughtfully implemented, there are very few effective frameworks that guide mathematics educators to identify pre-service teachers' SCK. Some existing measures of pre-service teachers' mathematics content knowledge (including SCK) such as those suggested by Beswick and Goos (2012) and Gallant and Mayer (2012) have provided sound profiles in this area. However, validated and published measures to inform educators and researchers of how good the pre-service teachers' SCK is are rare.

To address this research gap, this paper reports an effort to conceptualise and develop the construct of SCK. We used existing research to develop a framework which extends the current notion of SCK. The framework has been trialled and evaluated with a group of pre-service teachers. While our research is an ongoing work, we choose to contribute our thoughts by sharing this framework in this paper because "our effort might be instructive to others trying to conceptualize, identify, measure, and ultimately improve teachers' Pedagogical Content Knowledge" (Hill, Ball & Schilling, 2008; p.373). Our paper aims to conceptualise the construct of SCK through elaborating the theoretical and empirical basis. The research question for this framework trial is whether the proposed framework enables researchers to identify the construct of SCK in the pre-service teachers' responses to a written test which examine their SCK. Ultimately, we strive to theorise and develop a more viable conceptual framework of teachers' *Specialised Content Knowledge*.

Identifying the components of Specialised Content Knowledge

Building on Ball et al.'s work (2008), in a study, Lin, Chin and Chiu (2011) suggested three elements of SCK, namely, *Explanation* (i.e., how to provide mathematical explanations for common rules and procedures), *Representation* (i.e., how to choose, make and use mathematical representations effectively and accurately), and *Justification* (i.e., justify whether the numerical answer is correct). We agree with Lin et al. that providing an accurate mathematical explanation, using an appropriate mathematical representation and justifying an answer are key elements of SCK, but argue that the notions of *Explanation* and *Representation* should be further elaborated if they are to be used effectively in teacher education programs. It is necessary to identify the possible components for each of the elements and to delineate relationships between them. The following section will discuss the proposed constructs of these elements and their significance.

Explanation

In the past few decades, conceptual and procedural knowledge tend to be dichotomised in Western mathematics education (Lai & Murray, 2015; Newton, 2008). However, some studies (such as Hiebert & Wearne, 1986; Watkins & Biggs, 2001) have reported that mathematical knowledge may not always be easily separated according to this dichotomy. There has been ongoing debate about the developmental relationship between conceptual and procedural knowledge (Rittle-Johnson, Siegler, & Alibali, 2001). Shulman (1986) argued that mathematical competence requires a linking of conceptual and procedural knowledge. Rittle-Johnson et al. (2001) have posited that procedural and conceptual understanding do not develop independently but rather iteratively, with gains in one leading to gains in the other, which in turn trigger new gains in the first. In this representation, the two kinds of knowledge exist in an interlocking process and are complementary to each other. Likewise, Hiebert et al. (1996) also pointed out that students integrated conceptual knowledge with their procedural skills when developing strategies for constructing new procedures. Developing one's procedural knowledge in a domain is important for improving one's conceptual knowledge in that domain, just as developing conceptual knowledge is essential for generation and selection of appropriate procedures (Rittle-Johnson et al., 2001).

Building on these ideas, we argue that, in order to fully facilitate pre-service teachers' learning to teach, the notion of *Explanation* under SCK should include procedural explanations and conceptual explanations. Conceptual explanations are further divided into conceptually mathematically-based explanations and conceptually practically-based explanations (Levenson, Tsamir, & Tirosh, 2010). Figure 1 shows the hierarchical construct of *Explanation*. Levenson et al. (2010) have pointed out that types of explanations used may vary in accordance with mathematical context, type of instructional activity, the aim of the explanation and student's age level. This construct will be further unpacked in the following section.

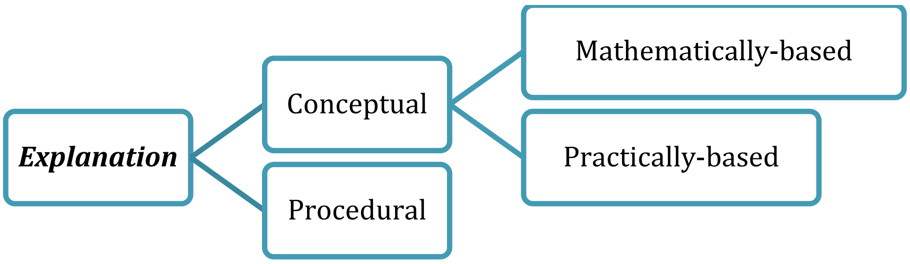


Figure 1. The hierarchical construct of *Explanation*

Procedural and conceptual explanations. Procedural explanation involves explication of the rules and procedures (Hiebert & Wearne, 1986; Skemp, 1976) which describe the steps taken during a mathematical task (Fuchs, Fuchs, Hamlett, Phillips, Karns, & Dutka, 1997). Wearne and Hiebert (1988) describe procedural understanding as syntactic processes which mean developing symbol-manipulation procedures and routinizing the rules for symbols. Procedural knowledge is also seen as knowledge of operators that can be applied to reach certain goals (Baroody, 2003; Rittle-Johnson et al., 2001). In contrast to this, a conceptual explanation requires knowledge of the connections between related concepts/principles and their interrelations in a domain (Schneider & Stern, 2010). Wearne and Hiebert (1988) hold a similar view of conceptual explanations and describe them as semantic-based processes, meaning connections between symbols with referents and the development of rules. Conceptual knowledge is also an understanding of why a procedure works (Hiebert & Wearne, 1986) and of whether a procedure is legitimate (Bisanz & Lefevre, 1992). Hiebert (1992) concludes that conceptual knowledge is the knowledge that is rich in relationships but not rich in techniques for completing tasks, while procedural knowledge is rich in rules and strategies but not rich in relationships.

For example, most people understand a procedure: "add a zero" to the number when multiplying by ten and "move the decimal point one digit to the left" when dividing by ten (Lai & Ho, 2012). Not only do teachers need to know how to do this mathematics, they also need to unpack and explain to students why this procedural rule works (Ball et al., 2008). In order to unpack mathematical knowledge, in a way that goes beyond the kind of tacit understanding of place value needed by most people, teachers need to know the principle of the decimal system for the conceptual explanation (refer to Hiebert, 1992; p. 286).

In another example of procedural knowledge, most adults are proficient in executing an algorithm for the multiplication of fractions and know that the answer can be smaller than the original number (i.e., multiplicand) if the multiplier is less than one. However, not all adults are able to give a mathematical reason why the statement "multiplication always makes bigger" is only true for whole numbers and not for fractions and decimals, a common learning difficulty of primary students. In order to unpack this conceptual knowledge, teachers need to know that the principle of subdividing the two sides of a two-dimensional shape into equal strips is used to explain the mathematical concept for multiplication of one fraction by another fraction (Ocheful, 2013). As the area of a rectangle can be calculated by multiplying the length times the width, now consider a rectangle measuring 1 by 1 (Newmark & Lake, 1974). For example, *the operation* $\frac{1}{4} \times \frac{1}{3}$ can be represented in a rectangle as shown in Figure 2.

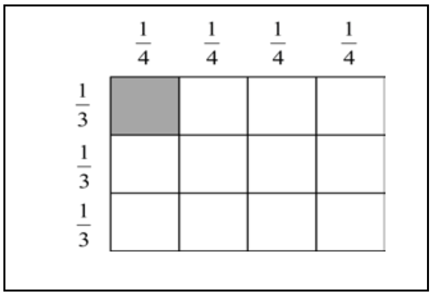


Figure 2. Subdividing two sides of a 2-dimensional shape for $\frac{1}{4} \times \frac{1}{3}$

Thus, the shaded portion of the diagram is measured by $\frac{1}{4} \times \frac{1}{3}$ and the area is $\frac{1}{12}$. Some educators may critique that this explanation is too mathematically rigorous for primary students to fully understand, as it is based on purely mathematical definitions (Levenson et al., 2010). Instead, those explanations which involve daily context and manipulatives may help students acquire the concept. The following is an alternate explanation to multiplication of fractions - the idea of recursive partitioning (Izsak, 2008).

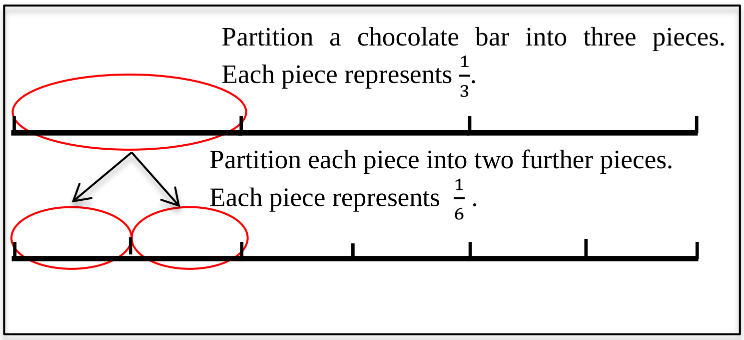


Figure 3. Recursive partitioning for $\frac{1}{2} \times \frac{1}{3}$

Izsak defined recursive partitioning as taking a partition of a partition. For example, we can consider a task of $\frac{1}{2} \times \frac{1}{3}$ as a result of taking $\frac{1}{2}$ of $\frac{1}{3}$ of a chocolate bar. Figure 3 shows the repeated partitioning. Thus, the pictures show clearly why, though the operation for $\frac{1}{2}$ of $\frac{1}{3}$ is multiplication, the outcome is $\frac{1}{6}$ which is smaller than $\frac{1}{3}$.

Sub-division of conceptual explanations - conceptually mathematically-based explanations and conceptually practically-based explanations. Levenson et al. (2010) further divide conceptual explanations into conceptually mathematically-based explanations and conceptually practically-based explanations. A conceptually mathematically-based explanation is "based on mathematical definitions or previously learned mathematical properties, and often uses mathematical reasoning" (Levenson et al., 2010, p. 346). The principle of subdividing two sides of a 2-dimensional shape for multiplication of fractions outlined earlier is an example of conceptually mathematically-based explanation. Though this type of explanation is based solely on mathematical notions, they are not necessarily rigorous (Levenson, Tirosh, & Tsamir, 2004; Levenson et al., 2010). For example, a conceptually mathematically-based explanation for whole number multiplication is repeated addition, that is the meaning of 3×2 can be interpreted as "2 and another 2 is 4 and another 2 is 6" (Levenson et al., 2004; p. 243) or simply 2+2+2=6. A conceptually practically-based explanation is one which uses "daily contexts and/or manipulatives to 'give meaning' to mathematical expressions" (Levenson et al., 2010, p. 345). Many explanations of this type include visual aids, stories and concrete objects (Levenson et al., 2004). The recursive partitioning for $\frac{1}{2} \times \frac{1}{3}$ outlined earlier is an example of conceptually practically-based explanation. Another example, the meaning of 3×2 can be represented by a story: "I have 3 sets of 2 pencils and therefore I have altogether 6 pencils." Alternatively, 3×2 can be represented in an array diagram.

One may argue that teachers should bring rigorous mathematical explanation such as conceptually mathematically-based explanation into classrooms as early as possible (Fischbein, 1987). However, other scholars such as Levenson et al. (2004) and Raman (2002) have pointed out that primary school students may be too young for rigorous mathematical explanations but can be convinced by daily life explanations such as conceptually practically-based explanation. Subsequently, Levenson at al. (2010) suggest a continuum of explanations, that is "beginning with practically-based explanations that use every day concrete objects, proceeding to semi-structured manipulatives, models, and generalized visual arguments, continuing to mathematically-based explanations and ending with formal explanations" (p. 349). Building on this idea, this paper supports Raman's (2002) position that both conceptually mathematically-based and practically-based explanations are needed and useful. In conclusion, to properly equip pre-service teachers to teach mathematics effectively, this paper argues that conceptually mathematically-based and conceptually practically-based explanations should be included in the notion of conceptual explanation.

Thus in summary, three types of explanation, namely, procedural explanation, conceptually mathematically-based explanation and conceptually practically-based explanation are proposed to be included in the notion of SCK.

Representation

Knowledge about the world may exist, or be represented, in different forms (Anderson, 2010). Representations of a concept might be in terms of a set of propositions or in a visual image. Representation of a procedure might be in a list of steps. Overall, representation is defined as a process and/or a product that can capture a mathematical concept or a relationship in some forms (NCTM, 2000). Different studies (such as Azevedo, van Dooren, Clareboot, Elen, & Verschaffel, 2009; de Freitas & Sinclair, 2012) have reported that multiple representations are essential for the construction processes of mathematical understanding. Gagatsis and Shiakalli (2004) argue that one of the criteria for determining a person's understanding of a mathematical concept entails "the ability to recognise an idea which is embedded in a variety of qualitatively different representational systems" (p. 645). Likewise, teachers' ability to use multiple representations for explaining mathematical concepts is essential "in supporting students' understanding of mathematical concepts and relationships" (NCTM, 2000; p. 67). Font, Godino and D'Amore (2007) have pointed out that representation can refer to any mathematical activities, cultural and cognitive productions and also those related to the world that surrounds us. Then, the question arises as to what types of representations are relevant to classroom teaching and students' learning.

Researchers such as Usiskin (1987), Hegarty and Kozhevnikov (1999) and van Garderen (2006) argue that visual representations (such as images, pictures, concrete objects, gestures, diagrams and graphs) are important in mathematics education and enhance understanding in many areas of mathematics. Similarly, Presmeg (1999) and Pape and Tchoshanov (2001) point out that the use of particular modes of representation such use of visual representations and concrete objects leads to improvement of primary students' mathematical abilities and development of their problem solving skills. Their argument aligns with Bruner's (1966) learning theory that through exploration of different forms of representation - concrete materials, pictures and symbols - students become more competent in capturing the abstract symbolic representations of mathematical ideas and operations. Thus, this paper proposes to include visual representation as one of the components of Representation.

In addition to visual representations, non-visual representations such as daily life situations, stories, symbols and metaphors play equivalent roles to visual aids in developing students' understanding of mathematical concepts. Building on this idea, we argue that the notion of representation should include visual representation and non-visual representation, all of which are commonly used in conceptual explanations in mathematics classrooms. In considering that procedural explanation primarily involves merely the manipulation of rules and procedures in symbolic forms, there is no further differentiation of representation in this type of explanation. As a result, two different representations, namely visual representation and non-visual representation for conceptually mathematically-based and practically-based explanation, are proposed to be included in the notion of SCK.

The extended SCK framework

The ability to justify a mathematical idea is determined by the person's capacities in providing (1) correct numerical answers, (2) accurate explanation(s) and (3) using appropriate representation(s). In this framework, *Justification* refers to the correctness of the answer. For instance, a pre-service teacher may provide a correct answer with inaccurate or partially correct explanation. As a result, we developed a model to describe the relationship between *Justification*, *Explanation* and *Representation* under the notion of SCK as shown in Figure 4.

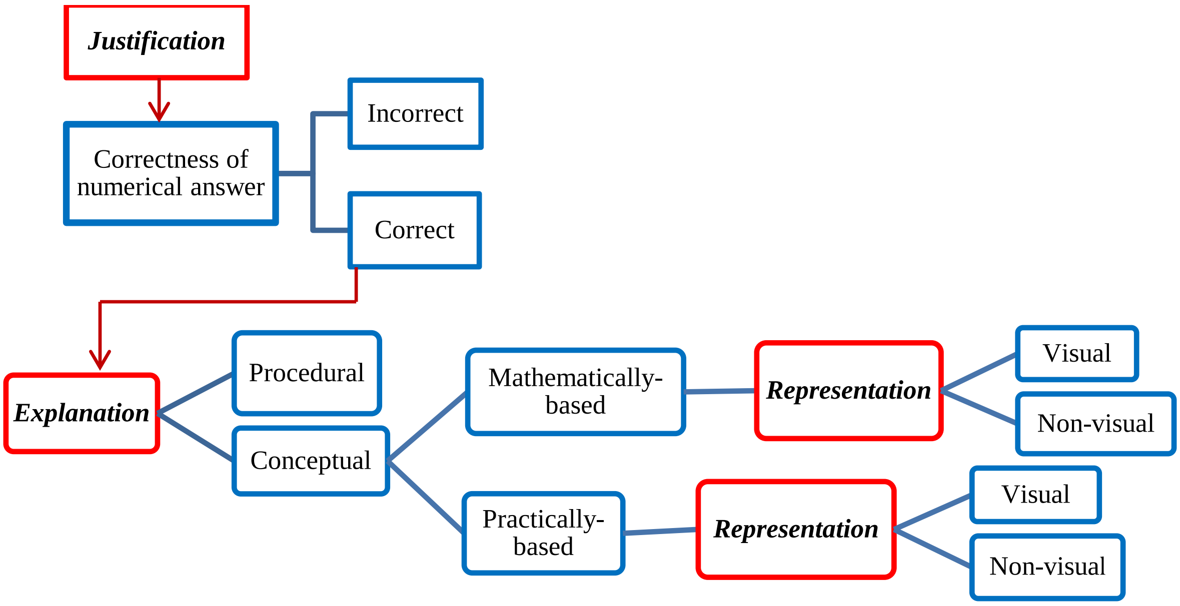


Figure 4. Framework for the extended notion of SCK (Modified from Lai & Ho, 2012)

Trial of the framework

The guiding question for this framework trial was whether the proposed framework enables researchers to identify the construct of SCK in the pre-service teachers' responses to a written test which examine their SCK. A crucial stage in our development of constructs in SCK was moving the conceptualisation of the framework into practice; that is trialling the framework with a cohort of pre-service teachers. In this section, we describe the participants and their background, the process of the trial and the scoring method.

Participants

In a regional university in Australia, the first year mathematics pedagogy subject was designed to develop pre-service teachers' SCK. The subject provided pre-service teachers with concrete links between the procedures, concepts and representations related to those procedures (Newton, 2008). Ninety first year Bachelor of Education (Primary) pre-service teachers enrolled in this subject and sat for a mathematics test at the end of the subject in the second semester. This mathematics test formed part of the assessment criteria that these pre-service teachers needed to fulfil in their degree course. With appropriate consent, the students' responses to the written test provided the empirical basis for a trial of the proposed framework of SCK.

Procedures

The pedagogy subject included a one-hour lecture and a two-hour workshop per week for 14 weeks. In each week, the learning content focused on one big topic such as addition and subtraction of whole number, fractions, perimeter, 2D shapes and data handling. The specialised content knowledge, including procedural and conceptual knowledge, of different topics was covered in the lectures. In the workshops, different ways of explaining mathematical concepts (i.e., procedural, conceptually mathematically- based and practically-based explanations) using a range of representations (i.e., visual and non-visual) were demonstrated and discussed. Varieties of authentic hands-on activities/tasks were provided for the pre-service teachers to consolidate their understanding of different types of explanations in different forms of representation.

In the first week of the semester, pre-service teachers were provided with 50 tasks that presented various scenarios of primary school students' answers for different mathematics topics. Each of the tasks required the pre-service teachers to address the primary school students' misconceptions, demonstrate their basic mathematics skills and give explanations for the underlying mathematical concepts in different forms of representation involved in the task. Lectures and workshops provided sufficient details of and discussions for the related mathematical concepts. For instance, the pre-service teachers had worked directly with some of the 50 tasks within workshops, where they were presented with various examples of conceptually-based explanations in different forms of representation. The pre-service teachers were well informed of the requirements of the mathematics test including providing (1) correct numerical answers and (2) both procedural and conceptual explanations in (3) different forms of representation for all what and why questions which signified the demand of different types of explanation in different forms of representation. The pre-service teachers were required to work outside of the lecture and workshop time to ensure that they were competent to respond appropriately to all of the tasks.

The mathematics test consisted of 10 tasks which were selected randomly from the original 50 tasks provided to the participants at the beginning of the semester. The tasks that have been attempted in the tutorials were excluded from the test. The pre-service teachers sat for the test in the last week of the semester.

Scoring method

Based on the earlier discussion of the proposed constructs of *Specialized Content Knowledge* as shown in Figure 4, each pre-service teacher's response to the task was analysed in the following way:

1. *Justification* of the numerical answer: Each response to the task was first checked for *Justification*, that is to decide whether or not the numerical answer was correct.
2. Check for *Explanation*: Next, the response was checked for *Explanation*. Here the decision made was whether the explanation was procedural or conceptual. A procedural explanation was an explanation based on how to execute the algorithms. A conceptual explanation involved knowing the underlying mathematical reasons for the algorithms. A conceptual explanation was then further analysed as to whether it was a mathematically-based explanation or a practically-based explanation.
3. Check for *Representation*: Responses that were identified as conceptually mathematically-based and/or conceptually practically-based explanations were further analysed for the form of *Representation*, visual and/or non-visual representation.

To establish the validity and reliability of the coding process, the first author and a mathematics education researcher who taught the subject each independently coded the 90 pre-service teachers' responses with 85% agreement.

This paper focuses on trialling the framework on **two** items of the mathematics test: **multiplication of fractions** and **multiplication of decimals** as in the following:

Fraction Task:

> Cameron is working on the following problem:
> There is 3/4 of a pizza left after the party. 1/3 of the left-over are given to Sarah to take home. What fraction of a pizza does Sarah take home?
> You hear Cameron say "One-third of three-quarters; that's the same as one-third times three-quarters…" Peers of Cameron are very puzzled about his answer.
> What is the answer to the problem? How could you explain to the class?

Decimal Task:

> Susan got the answer 0.9 to the question 0.3×0.3.
> (a) Is the answer correct? Why?
>
> (b) How could you explain to your class?

The items put emphasis on "*how could you explain the concept to the class,*" a key part of our conceptualisation of SCK which is distinct from general mathematics content knowledge. We realised that a teacher who holds strong knowledge in general mathematics content knowledge may not necessarily have sufficient knowledge for explaining a concept. Fraction and decimal test items were selected to report in this study because they are considered to be the most complex mathematical domain in primary school mathematics (Ball, 1990). The limitations/weaknesses for confining the trial to the Number strand will be discussed later in this paper.

Results

The purpose of this section is not to perform validation work. Instead, we see the pre-service teachers' responses to the written test providing data suitable for investigating the involvement of and utility of our proposed additional components of SCK. This analysis allowed insight into our central question: whether the proposed framework enables researchers to identify the construct of SCK in the pre-service teachers' responses to a written test which examine their SCK.

The following section discusses the results of using the proposed constructs to analyse the pre-service teachers' responses to the two items - fraction and decimal items. Tables 1, 2, 3 and 4 report the results related to the components of SCK. Table 1 shows the frequency and percentage of *Correctness of the numerical answers* and number of *Explanation* types. Table 2 shows the frequency and percentage correct of different types of *Explanation*. Tables 3 and 4 show the frequency and percentage correct of different types of *Explanation* in different forms of *Representation* that the pre-service teachers used in their self-generated responses to the fraction and decimal items. Table 5 displays the overall performance in providing correct answers, correct explanations and correct representations. This aspect of the data depicts a general picture for the central questions outlined earlier.

Table 1
Frequency and percentage of Justification and types of Explanation for fraction and decimal items (N=90)

		Item	Frequency	Percent
Justification (Correctness of the numerical answer)	incorrect	Fraction	16	17.8
	correct		74	82.2
	incorrect	Decimal	29	32.2
	correct		61	67.8
Explanation	No type	Fraction	24	26.7
	1 type		37	41.1
	2 types		23	25.6
	3 types		6	6.7
	No type	Decimal	28	31.1
	1 type		49	54.4
	2 types		13	14.4
	3 types		0	0

Table 2
Frequency and percentage correct of procedural explanation, conceptually mathematically-based explanation and conceptually practically–based explanation for fraction and decimal items (n=90)

Type of *Explanation*	**Item**	**Frequency correct**	**Percentage correct**
Procedural	Fraction	45	50
	Decimal	32	35.6
Conceptually Mathematically-Based	Fraction	15	16.7
	Decimal	31	34.4
Conceptually Practically-Based	Fraction	42	46.7
	Decimal	11	12.2

Table 3
Frequency and percentage correct of Representation for conceptually mathematically-based explanation for fraction and decimal items (n=90)

Representation	**Item**	**Frequency correct**	**Percentage correct**
Non-visual	Fraction	2	2.2
	Decimal	31	34.4
Visual	Fraction	13	14.4
	Decimal	6	6.7

Table 4
Frequency and percentage correct of Representation for conceptually practically-based explanation for fraction and decimal items (n=90)

Representation	**Item**	**Frequency correct**	**Percentage correct**
Non-visual	Fraction	19	21.1
	Decimal	6	6.7
Visual	Fraction	34	37.8
	Decimal	7	7.8

Table 5
The overall performance in providing correct answers, correct explanations and correct representations (n=90)

Overall performance	**Item**	***f***	**%**
[*Justification of Correctness*]: No/Incorrect answer, [*Explanation*]: No/Incorrect, [*Representation*]: No/Incorrect	Fractions	14	15.6
	Decimals	26	28.9
[*Justification of Correctness*]: Correct answer only [*Explanation*]: No/Incorrect, [*Representation*]: No/Incorrect	Fractions	10	11.1
	Decimals	3	3.3
[*Justification of Correctness*]: Correct answer, [*Explanation*]: Procedural, [*Representation*]: No	Fractions	19	21.1
	Decimals	25	27.8
[*Justification of Correctness*]: Correct answer, [*Explanation*]: Conceptually mathematically-based [*Representation*]: Visual/ Non-Visual; **OR**	Fractions	12	13.3
[*Justification of Correctness*]: Correct answer, [*Explanation*]: Conceptually practically-based, [*Representation*]: Visual/Non-Visual	Decimals	18	20
[*Justification of Correctness*]: Correct answer, [*Explanation*]: Procedural and Conceptually mathematically-based, [*Representation*]: Visual/Non-Visual; **OR**	Fractions	23	25.6

[*Justification of Correctness*]: Correct answer, [*Explanation*]: Procedural and Conceptually practically-based, [*Representation*]: Visual/Non-Visual; **OR** [*Justification of Correctness*]: Correct answer; [*Explanation*]: Conceptually mathematically-based; [*Representation*]: Visual and non-visual; **OR** [*Justification of Correctness*]: Correct answer; [*Explanation*]: Conceptually practically-based; [*Representation*]: Visual and non-visual	Decimals	11	12.2
[*Justification of Correctness*]: Correct answer, [*Explanation*]: Procedural and Conceptually mathematically-based, [*Representation*]: Visual and Non-visual; **OR** [*Justification of Correctness*]: Correct answer, [*Explanation*]: Procedural and Conceptually practically-based, [*Representation*]: Visual and Non-visual; **OR** [*Justification of Correctness*]: Correct answer, [*Explanation*]: Conceptually mathematically-based and Conceptually practically-based, [*Representation*]: Visual/Non-visual and Visual/Non-visual	Fractions	5	5.6
	Decimals	7	7.8
[*Justification of Correctness*]: Correct answer; [*Explanation*]: Procedural, Conceptually mathematically-based and Conceptually practically-based , [*Representation*]: Visual/Non-visual and Visual/Non-visual; **OR** [*Justification of Correctness*]: Correct answer; [*Explanation*]: Conceptually mathematically-based and Conceptually practically-based , [*Representation*]: Visual and Non-visual, and Visual/Non-visual; **OR** [*Justification of Correctness*]: Correct answer; [*Explanation*]: Conceptually mathematically-based and Conceptually practically-based , [*Representation*]: Visual/Non-visual, and Visual and Non-visual	Fractions	7	7.8
	Decimals	0	0
[*Justification of Correctness*]: Correct answer; [*Explanation*]: Conceptually mathematically-based and Conceptually practically-based , [*Representation*]: Visual and Non-visual, and Visual and Non-visual	Fractions	0	0
	Decimals	0	0
[*Justification of Correctness*]: Correct answer; [*Explanation*]: Procedural, Conceptually mathematically-based and Conceptually practically-based , [*Representation*]: Visual and Non-visual, and Visual and Non-visual	Fractions	0	0
	Decimals	0	0

Construct of Justification

For the *Justification*, about 82% of pre-service teachers provided correct answers for $\frac{1}{3} \times \frac{3}{4}$ in the fractions item and over 67% correctly responded that 0.9 is not an answer for 0.3 x 0.3 but 0.09 for the decimal item. However, not all pre-service teachers who answered correctly were also able to provide appropriate explanations for their answers as discussed in the following sections.

Construct of Explanation

For the fraction item, the results indicated that slightly less than three quarters of the pre-service teachers provided **at least one type of explanation**. About 41%, 26% and 7% of the pre-service teachers correctly gave one type, two types and three types of explanations respectively. Among the pre-service teachers who provided one type of explanation, about 49% gave correct procedural explanations, 46% conceptually practically-based explanations and 5% conceptually mathematically-based explanations. Among the pre-service teachers who gave two types of explanation, about 17% correctly provided conceptually mathematically-based and practically-based explanations, 22% procedural and conceptually mathematically-based explanations, and 57% procedural and conceptually practically-based explanations. More pre-service teachers provided practically-based explanations than mathematically-based explanations for conceptual explanations in this fraction task.

For the decimal item, the results showed that slightly over two-thirds of the pre-service teachers correctly provided **at least one type of explanation**. About 54 % and 14% of the pre-service teachers gave one type and two types of explanation respectively. However, no one pre-service teacher provided all three types of explanation. Among the pre-service teachers who correctly provided one type of explanation, about 47% gave correct procedural explanations, 8% conceptually practically-based explanations and 45% conceptually mathematically-based explanations. Among the pre-service teachers who gave two types of explanation, about 30% provided correct conceptually mathematically-based and practically-based explanations, 46% procedural and conceptually mathematically-based explanations, and 23% procedural and conceptually practically-based explanations. Similar to the fraction item, the number of pre-service teachers who chose to provide procedural explanations was similar to that of pre-service teachers who chose to give conceptual explanations. However, unlike the fraction item, more pre-service teachers chose to provide mathematically-based explanations than those who gave practically-based explanations for the decimal item.

Procedural Explanation

For the fraction item, half of the pre-service teachers provided correct procedural explanations. Many of the correct responses demonstrated direct multiplication of the digits in the numerators and in the denominators: $\frac{1}{3} \times \frac{3}{4} = \frac{1\times 3}{3\times 4} = \frac{3}{12}$ (refer to Figures 8 and 9 for example of written responses). These pre-service teachers correctly applied a whole number multiplication procedure and this may have been due to the small digits in the fractions. Among those pre-service teachers who did not provide correct procedural explanations, some misapplied algorithms such as cross multiplied, others found a common denominator and then kept that denominator in their product, and some flipped the multiplier as used for the division of fractions. The results were in line with the findings reported by Newton (2008), and Isiksal and Cakiroglu (2011). These kinds of procedural errors were identified as "algorithmically based mistakes" and explained by the fact that some people apparently have understood the rote algorithms needed to manipulate the symbols but soon forgot and mixed up the procedures (Isiksal & Cakiroglu, 2011). This error might stem from pure rote memorisation of the algorithms without understanding the underpinned concepts (Isiksal & Cakiroglu, 2011).

For the decimal item, the results indicated that about 35% of the pre-service teachers provided correct procedural explanations. Many of the correct responses were very straight forward reflecting a simple method of counting the number of decimal places in the multiplier and multiplicand, and then moving the decimal point in the product to get the corresponding number of decimal places. The following is a topical response for procedural explanation (refer to Figure 12 for this written response):

> *In multiplication of decimals, the number of digits to the right of the decimal place (added together) indicates how many places to the left of the answer, there should be 1d.p + 1d.p = 2p.d for* 0.3×0.3*, so 009 →0.09*

Many pre-service teachers who provided incorrect responses for procedural explanations were not even aware of the number of decimal places but just recognised that 3 times 3 was 9 and the product must be a decimal because it was a decimal multiplication. Consequently, they came up with 0.9 as their answers for 0.3 times 0.3.

Conceptually mathematically-based Explanation in visual and non-visual Representation

For the fraction item, many of the correct mathematically-based explanations were provided in visual representation using the principle of subdividing two sides of a 2-dimensional shape. Figure 5 illustrates a topical response.

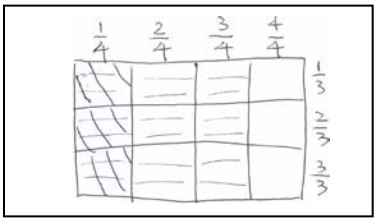


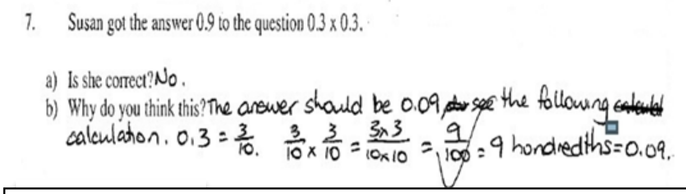
Figure 5. An example of response presented in conceptually mathematically-based explanation using the principle of subdividing two sides of a 2-dimensional shape in visual representation

Only two pre-service teachers chose to present the mathematically-based explanations in non-visual representation such as equations or words. The following shows a pre-service teacher's response:

> $\frac{1}{3}$ *of* $\frac{3}{4}$ *could be understood as* $\frac{3}{4}$ *was divided (by) 3. Even if it was divided by 3, for purposes of fractions, it would still end up multiplying by* $\frac{1}{3}$ *anyway.*

Another pre-service teacher (refer to Figure 10 for the written response) represented one-third of three quarters is one quarter in an equation of repeated addition $\frac{1}{4}+\frac{1}{4}+\frac{1}{4}$.

For the decimal item, among all the correct responses for conceptually mathematically-based explanations, many of the responses were presented in non-visual form. The pre-service teachers associated 0.3×0.3 with $\frac{3}{10}\times\frac{3}{10}$ which gave a product of $\frac{9}{100}$ and then translated $\frac{9}{100}$ to 0.09. It illustrated that some pre-service teachers made good connections between fractions and decimals. Figures 6 and 12 show examples of this type of response.



In print: The answer should be 0.09, see the following calculation, 0.3= $\frac{3}{10}$, $\frac{3}{10} \times \frac{3}{10} = \frac{3 \times 3}{10 \times 10} = \frac{9}{100}$ =9 hundredths =0.09

Figure 6. An example of conceptually mathematically-based explanation in non-visual representation for 0.3 × 0.3

Few pre-service teachers provided correct mathematically-based explanations in both visual and non-visual representations for 0.3 × 0.3. They applied the principle of subdividing two sides of a two-dimensional shape to the decimal item and presented the concept in similar diagrams as in the fraction item. Figure 7 demonstrates an example of this response. Unlike the response shown in Figure 5 for the fraction item, the pre-service teacher did not denote the dimension of each of the subdivisions (i.e., 0.1 or $\frac{1}{10}$).

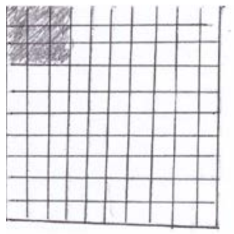


Figure 7. An example of response presented in conceptually mathematically-based explanation using principle of subdividing two sides of a 2-dimensional shape in visual representation for 0.3 × 0.3

Conceptually practically-based Explanation in visual and non-visual Representation

In fraction items, nearly half of the pre-service teachers chose to use daily life examples and practical explanations. Among those pre-service teachers who provided correct practically-based explanations, about two-third of the responses were given in a visual representation. About one-third of the responses were presented in non-visual representation. Figure 8 demonstrates a response of practically-based explanation using recursive partitioning presented in both visual and non-visual representations for $\frac{1}{3} \times \frac{3}{4}$. The pre-service teacher firstly divided the pizza into four equal parts with three parts shaded as shown in the first picture in Figure 8. The pre-service teacher then further divided the three shaded parts into another three equal parts and shaded them as shown in the second picture in Figure 8. The pre-service teacher concluded his/her drawing in a sentence $\frac{1}{3}$ of $\frac{3}{4}$.

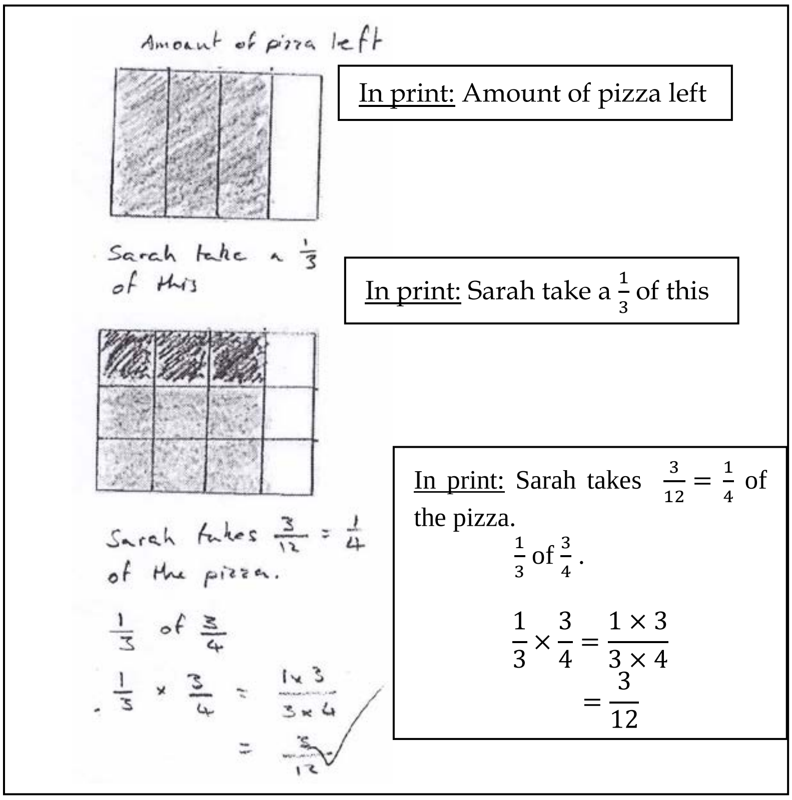


Figure 8. An example of procedural explanation and conceptually practically-based explanation using recursive partitioning in both visual and non-visual representation for multiplication of fraction

Instead of using recursive partitioning, some pre-service teachers used a concept of *sub-division* in visual representation for their conceptual practically-based explanation. Figures 9 and 10 demonstrate this point. In Figure 9, the pre-service teacher drew $\frac{3}{4}$ of the pizza which was the left over part. The pre-service teacher then shaded one part out of the three to indicate the portion that was taken home, which indicated $\frac{1}{3}$ of $\frac{3}{4}$ and concluded that "$\frac{3}{4}$ *of a pizza remaining,* $\frac{1}{3}$ *is taken. Therefore* $\frac{1}{4}$ *of the whole pizza was taken*." Figure 10 shows another example presented in both visual and non-visual representation. This pre-service teacher provided a similar diagram as in Figure 9 and replied in words (non-visual) that "$\frac{3}{4}$ *of a pizza is (the) left over which (is) divide(d) into three equal parts (and each part is) equal (to)* $\frac{1}{4}$*. So Sarah will take* $\frac{1}{4}$ *of the pizza home*."

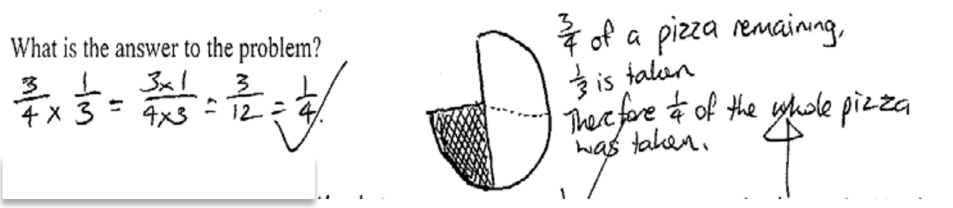


Figure 9. An example of procedural and conceptually practically-based explanations using sub-division in visual representation for multiplication of fractions

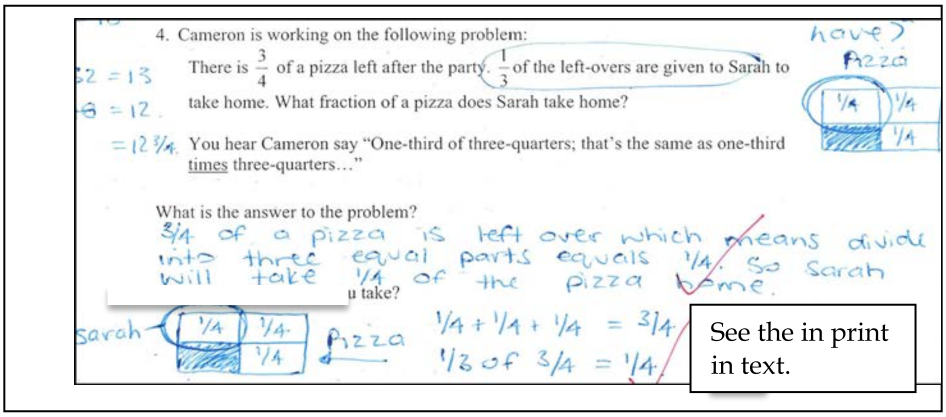


Figure 10. An example of conceptually mathematically-based explanation in non-visual representation and conceptually practically-based explanation in visual and non-visual representations

The following illustrates another pre-service teacher’s response in conceptually practically-based non-visual representation as well.

> *… takes $\frac{1}{3}$ of the $\frac{3}{4}$, that is divide the pizza into even numbers. In this case, divide the whole into 12 even parts to get 3 parts of the whole (i.e., 12 even parts) or $\frac{1}{4}$ of the whole.*

For the decimal item, not many pre-service teachers correctly provided conceptually practically-based explanations using the principle of recursive partitioning in visual form. Figures 11 and 12 illustrate these responses from two different pre-service teachers but the picture in Figure 11 is not totally correct.

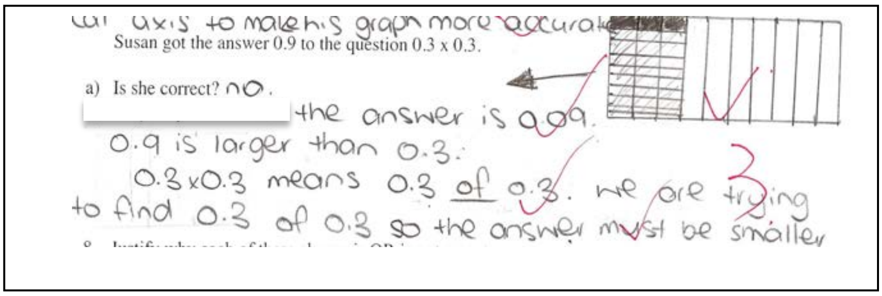


Figure 11. An example of conceptually practically-based explanation using recursive partitioning in visual and non-visual representation for 0.3 × 0.3

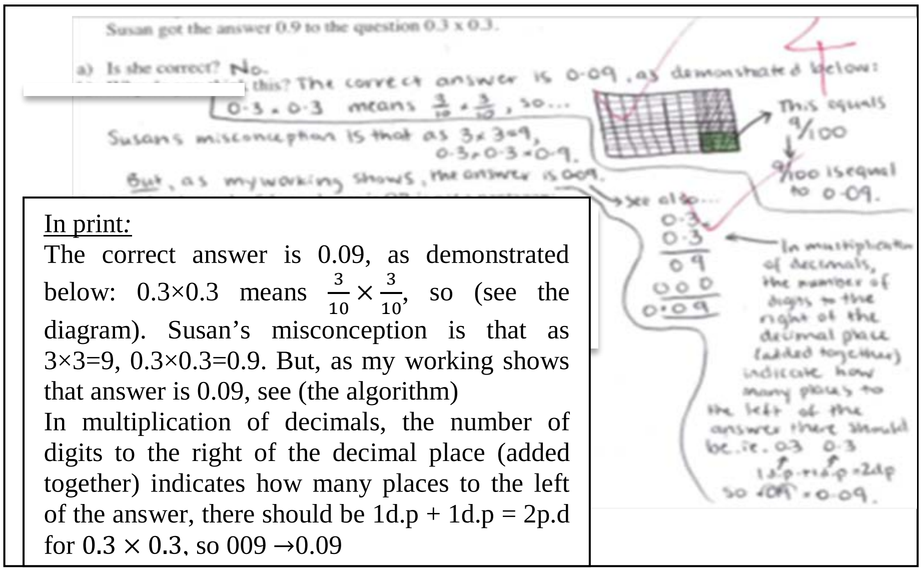


Figure 12. An example of procedural explanation, conceptually mathematically-based explanation in non-visual representation and conceptually practically-based explanation in visual representation for 0.3×0.3

Similar to the fraction item, in Figure 12 the pre-service teachers firstly divided a rectangle into 10 equal parts then shaded 3 parts; the shaded part represented 0.3. The pre-service teachers then further divided the 3 parts into another 10 equal parts and lastly shaded 3 parts; the darken portion indicated 0.3 of 0.3.

Only few pre-service teachers gave their conceptually practically-based explanations using recursive partitioning in non-visual form. The following illustrates this type of explanation. Another example is displayed in Figure 11.

> 0.3 × 0.3 *sometimes is easier to think of as 0.3 of 0.3. So the answer would have to be less than 0.3. The answer is 0.09.*

As reported earlier, for the fraction item only few pre-service teachers chose to use both types of conceptual explanations as shown in Figure 10. No pre-service teacher gave both types of conceptual explanations for the decimal items.

Discussion

No one would argue the fact that *Specialised Content Knowledge* (SCK) is essential to *Pedagogical Content Knowledge* for effective and quality mathematics teaching. However, the specific constructs that compose SCK remain underspecified. To address this research gap, this study used existing research to create a framework that extends the notion of SCK and trial the framework on a cohort of pre-service teachers by evaluating the results from a written test according to our created framework. This discussion is framed around the study research question: whether the proposed framework enables researchers to identify the construct of SCK in the pre-service teachers' responses to a written test which examine their SCK. We strive to develop a more viable conceptual framework of *Specialised Content Knowledge* that can advance the field.

The trial indicated that the proposed framework shows promise for revealing specific important constructs of SCK. The framework allowed for the identification of pre-service teachers' knowledge of fine-grained aspects of conceptual understanding including mathematically-based and practically-based explanations, as well as types of representations used to support student learning. The test items provided appropriate opportunities for pre-service teachers to respond according to their SCK. Analysis revealed details of individual understanding and patterns within a cohort. In general, an analysis of the test results with this framework has provided some themes of common strengths and weaknesses. For example, more pre-service teachers provided practically-based explanations than mathematically-based explanations for the fraction task. They preferred presenting their explanations in visual form. However in contrast, more pre-service teachers provided mathematically-based explanations for the decimal task. They preferred presenting their explanations in non-visual form. The results also have indicated that the majority of the pre-service teachers did not tend to provide a broad range of explanations to support the development of students' conceptual understanding. In fact, none of the pre-service teachers in the study provided all three types of explanations in the two forms of representations.

Overall, pre-service teachers appear to be more comfortable with particular explanation and representation types, despite being exposed to a broad range within lectures and workshops. It would be useful to survey the pre-service teachers with this framework at the beginning of program to ascertain their knowledge and conceptual understanding. This information could be used to differentiate workshop activities so that all pre-service teachers are provided with opportunities to strengthen and broaden their knowledge of explanations and representations, SCK in general.

The framework, used by multiple markers, allowed for identification of pre-service teachers' conceptual understanding and knowledge of appropriate explanation types in different forms of representation. This level of insight into pre-service teacher's knowledge has not been readily available through traditional testing and use of existing frameworks in literature review. Analysis indicated that using the proposed framework produced consistent interpretation of pre-service teacher responses. Being able to clearly identify pre-service-teachers use of different explanation types for teaching specific content has the potential to positively impact mathematics teacher education development. It may also empower pre-service teachers with practical information about their own pedagogical practices.

Furthermore the framework gave a new level of awareness about pre-service teacher preference for representations according to different content - in this case fractions and decimals. Being able to identify detailed components of SCK with a test of this nature has not been feasible on large scales. Such a framework has the potential to provide data about pre-service teacher's SCK in a new way for teacher educators.

Conclusion

This new approach has built on existing knowledge and research to fill a gap seen in existing frameworks. While this framework provided some useful insights into pre-service teachers' understanding of fractions and decimals, it is challenging for any framework to fully measure the complexity of SCK. Limitations noted through using this framework to analyse data highlighted the difficulty in distinguishing between the visual forms of mathematically-based and practically-based explanations. Some of the explanations provided by pre-service teachers indicated an overlap between these two types. Therefore, while all explanations were categorised according to these types, some may have been able to fit into both categories. Other limitations such as scalability and demography may also have adverse impact on the generality and applicability of the results.

By developing the framework for identifying pre-service teachers' SCK for teaching mathematics, this study attempts to contribute to identifying the construct of SCK for teaching by allowing insight into how knowledge is held by pre-service teachers and hence, to extend our notions of SCK. We suggest that three types of explanations (i.e., procedural explanations, and conceptual explanations including mathematically-based and practically-based explanations) and two forms of mathematical representations (i.e., visual and non-visual) of concepts should be introduced and discussed in mathematics pedagogy subjects in teacher education programs. We see the report of analysis in this paper as a first step in the development of a measurement of pre-service teachers' SCK. The data set was less than ideal because the analysis was only confined to number strands. Whether this framework is also workable for other strands such as space and geometry or measurement is not investigated within this study. Further study is required.

We believe that developing such a framework can also contribute to reshaping the traditional method of marking examinations of teacher education subjects. The advantage of the framework is that it allows the teasing out of the strengths and weaknesses in SCK, that is, by using the framework, lecturers would be able to find out which SCK components each of their pre-service classes is strong in (and weak in), thus enabling them to design their mathematics pedagogy courses accordingly. The framework was found to be effective for these reasons: (1) the framework is easy to follow, making marking straightforward and thus moderation of marks is now unnecessary in the case where there is more than one marker; (2) lecturers are informed about the types of explanations (procedural and conceptual) and the forms of mathematical representations (visual and non-visual) that their pre-service teachers have; and (3) lecturers are informed about their pre-service teachers' strengths and weaknesses of mathematics concepts.

In addition, use of this framework enables pre-service teachers to have clearer learning goals regarding the specialised content knowledge (SCK) for fractions and decimals. Further research could investigate the validity and reliability of the test items by using the framework, and the effectiveness of the proposed framework on pre-service teachers' learning.

References

Azevedo, N. A., van Dooren, W., Clareboot, G., Elen, J., & Verschaffel, L. (2009). Conceptualising, investigating and stimulating representational flexibility in mathematical problem solving and learning: A critical review. *ZDM the International Journal on Mathematics Education, 41*(5), 627-636.

Anderson, J. R. (2010). *Cognitive psychology and its implications (7th ed.)*. New York, NY, USA: W H Freeman/Times Books/ Henry Holt & Co Cognitive

Ball, D. L. (1990). The mathematical understandings that prospective teachers bring to teacher education. *Elementary School Journal*, *90*(4), 449-466.

Ball, D. L., Hill, H. C. & Bass, H. (2005). Knowing mathematics for teaching. *American Educator, 29*(3), 14-46.

Ball, D. L., Thames, M. H., Bass, H., Sleep, L., Lewis, J., & Phelps, G. (2009). A practiced-based theory of mathematical knowledge for teaching. In M. Tzekaki, M. Kaldrimidou, & H. Sakonidis, H. (Eds.), *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 95-98). Thessaloniki, Greece: PME.

Ball, D.L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, *59*(5), 389-407.

Baroody, A. J., (2003). The development of adaptive expertise and flexibility: The integration of conceptual and procedural knowledge. In A. J. Baroody & A. Dowker (Eds.), *The development of arithmetic concepts and skills: Constructive adaptive expertise* (pp.1-33). Mahwah, NJ: Routledge.

Baumert, J., Kunter, M., Blum, W., Brunner, M., Voss, T., Jordan, A., ... & Tsai, Y. M. (2010). Teachers' mathematical knowledge, cognitive activation in the classroom, and student progress. *American Educational Research Journal, 47*(1), 133-180.

Beswick, K. & Goos, M. (2012). Measuring pre-service teachers' knowledge for teaching mathematics. *Mathematics Teacher Education and Development, 14*(2), 70-90.

Betts P. (2011) Tensions of Mentoring Mathematics Teachers: Translating Theory into Practice. In: Schuck S., Pereira P. (eds), *What Counts in Teaching Mathematics. Self Study of Teaching and Teacher Education Practices*, *vol 11*. Dordrecht , The Netherlands: Springer,.Bisanz, J., & LeFevre, J. (1992). Understanding elementary mathematics. In J. I. D. Campbell (Ed.), *The nature and origins of mathematical skills* (pp. 113–136). Amsterdam, The Netherlands: Elsevier Science.

Bisanz, J. & LeFevre, J. (1992). Understanding elementary mathematics. In J. I. D. Campbell (Ed.), *The nature and origins of mathematical skills* (pp. 113–136). Amsterdam: North Holland Elsevier Science.

Bruner, J. S. (1966). *Towards a theory of instruction* (Vol. 59). Cambridge, Mass.: Belknap Press of Harvard University.

Cai, J., Mok, I. A., Reddy, V., & Stacey, K. (2016). International comparative studies in mathematics: Lessons for improving students' learning. In J. Cai, I. Mok, V. Reddy, & K. Stacey (Eds.), *International Comparative Studies in Mathematics* (pp. 1-36). New York, NY: Springer.

De Freitas, E., & Sinclair, N. (2012). Diagram, gesture, agency: Theorizing embodiment in the mathematics classroom. *Educational Studies in Mathematics, 80*(1-2), 133-152.

Fischbein, E. (1987). *Intuition in science and mathematics*. Dordrecht, The Netherlands: Reidel Publishing Company.

Font, V., Godino, J. D., & D'Amore, B. (2007). An onto-semiotic approach to representations in mathematics education. *For the Learning of Mathematics, 27*(2), 2-14.

Fuchs, L. S., Fuchs, D., Hamlett, C. L., Phillips, N. B., Karns, K., & Dutka, S. (1997). Enhancing students' helping behavior during peer-mediated instruction with conceptual mathematical explanations. *The Elementary School Journal, 97*(3), 223-249.

Gagatsis, A., & Shiakalli, M. (2004). Ability to translate from one representation of the concept of function to another and mathematical problem solving. *Educational Psychology: An International Journal of Experimental Educational Psychology, 24*(5), 645-657.

Gallant, A. & Mayer, D. (2012). Teacher performance assessment in teacher education: An example in Malaysia. *Journal of Education for Teaching, 38*(3), 295-307.

Hattie, J. A., & Donoghue, G. M. (2016). Learning strategies: a synthesis and conceptual model. Nature Partner Journal: Science of Learning, 1, 1–13.

Hegarty, M., & Kozhevnikov, M. (1999). Types of visual–spatial representations and mathematical problem solving. *Journal of Educational Psychology, 91*(4), 684-689.

Hiebert, J. (1992). Mathematical, cognitive, and instructional analyses of decimal fraction. In Leinhardt, G., Putnam, R. & Hattrup, R. A. (Eds), *Analysis of arithmetic for Mathematics teaching* (pp.283-321). Hillsdale, NJ : L. Erlbaum Associates.

Hiebert, J., Carpenter, T. P., Fennema, E., Fuson, K., Human, P., Murray, H., Olivier, A., & Wearne, D. (1996). Problem solving as a basis for reform in curriculum and instruction: The case of mathematics. *Educational researcher, 25*(4), 12-21.

Hiebert, J. & Wearne, J. (1986). Procedures over concepts: The acquisition of decimal number knowledge. In J, Heibert (Eds), *Conceptual and procedural knowledge: The case of Mathematics* (pp.199-223). Hillsdale, NJ : L. Erlbaum Associates.

Hill, H. C., Ball, D. L., & Schilling, S. G. (2008). Unpacking Pedagogical Content Knowledge: Conceptualizing and Measuring Teachers' Topic-Specific Knowledge of Students. *Journal for Research in Mathematics Education, 39*(4), 372-400.

Hill, H. C., Rowan, B., & Ball, D. L.. (2005). Effects of teachers' mathematics knowledge for teaching on student. *American Educational Research Association*, *42*(2), 371-406.

Isiksal, M. & Carkiroglu, E. (2011). The nature of prospective mathematics teachers' pedagogical content knowledge: the case of multiplication of fractions. *Journal of Mathematics Teacher Education, 14*(3), 213-230.

Izsak, A. (2008). Mathematical knowledge for teaching fraction multiplication. *Cognition and Instruction, 26*(1), 95-143.

Krainer, K., Hsieh, F. J., Peck, R., & Tatto, M. T. (2015). The TEDS-M: Important Issues, Results and Questions. In S. J. Cho (Ed.), *The Proceedings of the 12th International Congress on Mathematical Education* (pp. 99-121). Cham, Switzerland: Springer.

Lai, M. Y., & Ho, S. Y. (2012). Preservice teachers' specialized content knowledge on multiplication of decimals. In *12th International Congress on Mathematics Education* (p. 4616-4625). Seoul, South Korea: ICME.

Lai, M. Y., & Murray, S. (2015). Hong Kong grade six students' performance and mathematical reasoning in decimal tasks: Procedurally based or conceptually based? *International Journal of Science and Mathematics Education, 13*(1), 123-149.

Levenson, E., Tirosh, D., & Tsamir, P. (2004). Elementary school students' use of mathematically-based and practically-based explanations: The case of multiplication. In M. Hoines, & A. Fuglestad (Eds.), *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 241-248). Bergen, Norway: PME.

Levenson, E., Tsamir, P. & Tirosh, D. (2010). Mathematically based and practically based explanations in the elementary school: teachers' preferences. *Journal of Mathematics Teacher Education, 13*(4), 345–369

Lin, Y. C., Chin, C. & Chiu, H.Y. (2011). Developing an instrument to capture high school mathematics teachers' specialized content knowledge: An exploratory study. In Ubuz, B. (Ed.), *Proceedings of the 35th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 353). Ankara, Turkey: PME.

National Council of Teachers of Mathematics. (2000). *Principles and standards for schools mathematics.* Reston, Va: NCTM.

Newmark, J., & Lake, F. (1974). *Mathematics as a second language*. Menlo Park, Calif.: Addison-Wesley Publishing Company.

Newton, K. J. (2008). An extensive analysis of preservice elementary teachers' knowledge of fractions. *American Education Research Journal, 45*(4), 1080-1110.

Ocheful, A. S. (2013). Development of the concept of multiplication with fraction. *Universal Journal of Education and General Studies, 2*(3), 79-83.

Pape, S. J., Tchoshanov, M. A. (2001). The role of representation(s) in developing mathematical understanding. *Theory into practice, 40*(2), 118-127.

Presmeg, N. C. (1999, October). On visualization and generalization in mathematics. In F. Hitt & M. Santos (Eds.), *Proceedings of the twenty first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp.151-155). Columbus, OH.: Clearinghouse for Science, Mathematics, and Environmental Education.

Raman, M. (2002). Coordinating informal and formal aspects of mathematics: Student behaviour and textbook message. *Journal of Mathematical Behavior, 21*(2), 135-150.

Rittle-Johnson, B., Siegler, R. S., & Alibali, M. W. (2001). Developing conceptual understanding and procedural skill in mathematics: An iterative process. *Journal of Educational Psychology, 93*(2), 346-362.

Schneider, M., & Stern, E. (2010). The developmental relations between conceptual and procedural knowledge: a multimethod approach. *Developmental psychology, 46*(1), 178.

Skemp, R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching*, *77,* 20-26.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, *15*(2), 4-14.

Silverman, J. & Thompson, P. W. (2008). Toward a framework for the development of mathematical knowledge for teaching. *Journal of Mathematics Teacher Education, 11*(6), 499-511.

Usiskin, Z. (1987). Resolving the continuing dilemmas in school geometry. In M. M. Lindquist & A. P. Shulte (Eds.), *Learning and teaching geometry K-12* (pp. 17-31). Reston, VA: National Council of Teachers of Mathematics.

van Garderen, D. (2006). Spatial visualization, visual imagery, and mathematical problem solving of students with varying abilities. *Journal of learning disabilities, 39*(6), 496-506.

Watkins, D. A. & Biggs, J. B. (2001). The paradox of the Chinese learner and beyond. In D. A. Watkins & J. B. Biggs (Eds.), *Teaching the Chinese learner: Psychological and pedagogical perspectives* (pp. 3-23). Melbourne, Australia: ACER.

Wearne, D.& Hiebert, J. (1988). A cognitive approach to meaningful mathematics instruction: Testing a local theory using decimal numbers. *Journal for research in mathematics education, 19*(5), 371-384.

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