Secondary Practising Teachers' Professional Noticing of Students' Thinking About Pattern Generalisation

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In this article we describe secondary school practising teachers' professional noticing expertise, which includes (a) attending to the details of students' written or verbal responses, (b) interpreting students' mathematical understandings, and (c) deciding how to respond to students based on their understandings, with a focus on algebraic-pattern generalisation. Quantitative results indicated that the majority of teachers in our study provided evidence that they could attend to the mathematical details of students' thinking. However, the practising secondary school teachers provided less evidence of interpreting students' understandings, and even less of deciding how to respond to students based on those understandings. We present qualitative trends to help mathematics professional developers prepare for the teachers they will support and discuss how these trends might influence work with secondary school practising teachers on noticing student thinking in pattern-generalisation tasks.

Keywords Professional Noticing · Students' Algebraic Thinking · Secondary School Practising Teachers · Figural-Pattern Generalisation[1]

For more than two decades, the mathematics education community has recognised that instruction to respond to and build on student thinking is potentially beneficial for both students and teachers (Franke, Carpenter, Levi, & Fennema, 2001; Gearhart & Saxe, 2004; Kazemi, Franke, & Lampert, 2009; National Council of Teachers of Mathematics, 2014; Sowder, 2007; Wilson & Berne, 1999). However, implementing such instruction is challenging for many reasons. For one, this type of instruction requires students to actively engage with the teacher and with one another. With more students participating and sharing their thinking, teachers

must sift through an overwhelming amount of information to effectively respond to and support their students (Jacobs & Empson, 2016; Kazemi et al., 2009). These acts of filtering and making sense of classroom events have been conceptualised as acts of *teacher noticing* (for a summary, see Sherin, Jacobs, & Philipp, 2011). In classrooms where students regularly share ideas, the skill of noticing is a crucial part of effective teaching. Thus, we believe that teacher noticing merits further investigation.

Unlike other researchers who have investigated group differences in, changes in, or the effects of a particular intervention on teachers' professional noticing expertise, we investigate teachers' initial professional noticing expertise (cf. Lesseig, Casey, Monson, Krupa, & Huey, 2016; Simpson & Haltiwanger, 2016). Reasoning behind this decision has been summarised as follows:

> Just as teachers need to first determine what children understand so that they can use that understanding as a starting point for instruction, we argue that professional developers can use an understanding of teachers' reasoning in deciding how to respond to inform their professional development. (Jacobs, Lamb, Philipp, & Schappelle, 2011, p. 111)

In essence, we focus on providing mathematics professional development leaders with information about teachers from which they can build. In particular, our findings provide a window into practising secondary school teachers' sense-making and decision-making skills in the mathematical content-domain of pattern generalisation.

Background

Professional noticing of students' mathematical thinking is an expertise consisting of three component skills (Jacobs, Lamb, & Philipp, 2010). When a teacher engages with a student's thinking expressed in verbal or written form, the teacher

1. attends to the mathematical details of the student's explanation;
2. interprets the student's mathematical thinking; and
3. decides how to respond to that student on the basis of the student's thinking.

In the third component skill, professional noticing differs from other characterisations of teacher noticing, those focused solely on attending (e.g., Star & Strickland, 2008), or on attending and interpreting (e.g., Sherin & van Es, 2005). For an extensive examination of the affordances of this third component-skill of professional noticing, see Jacobs et al. (2011).

Elementary Versus Secondary Teachers[2]

We found no studies of practising secondary teachers' *professional* noticing of students' mathematical thinking[3] (i.e., attending, interpreting, and deciding how to respond). Most studies of professional noticing were focused on elementary rather than secondary school teachers (e.g., Schack et al., 2013), and in these studies participants were prospective teachers (e.g., Ding & Dominguez, 2016; Lesseig et al., 2016; Simpson & Haltiwanger, 2016). Hence, as a field we lack

[2] *Elementary* school teachers teach students aged 5–10 and *secondary* school teachers teach students ages 11–18.

[3] Some researchers (e.g., Fernandez, Llinares, & Valls, 2012) collected data on all three component skills but shared findings on only the first two. We did not consider such studies to be of professional noticing.

documentation regarding the degree to which practising secondary school teachers professionally notice students' mathematical thinking.

We argue that extrapolating results from studies of elementary school teachers' professional noticing expertise to secondary school teachers is inappropriate for two main reasons. First, elementary and secondary school teachers have differing experiences with students' mathematical thinking. Whereas in most elementary schools, one teacher has a class of 30 students, most secondary schools are departmentally organised whereby teachers teach one subject to several classes (Blatchford, Basset, & Brown, 2011; Ferguson & Fraser, 1998). Consequently, elementary school teachers spend a great deal of time with the same students but not with the same subject, the opposite of secondary school teachers' experiences. Second, elementary and secondary school teachers tend to differ in their orientations toward teaching. For example, Weinstein (1989) found that secondary school preservice teachers cited *strong mathematical knowledge* most frequently as a good teacher quality, whereas elementary school preservice teachers cited *capacity for caring*. Doig, Groves, Tytler, and Gough (2005) found similar results: on average, secondary school teachers felt more confident in supporting mathematical understandings and less confident in supporting a culture of respect than elementary school teachers. These differences led us to wonder whether professional noticing expertise may be influenced differentially by elementary school teachers' predominant focus on students versus secondary school teachers' predominant focus on mathematics.

For these reasons we address this gap in the literature and investigate practising secondary school teachers' professional noticing of students' mathematical ideas. In addition, we recognise that the difference between secondary and elementary levels is not the only salient difference to consider; teachers' professional noticing expertise appears different across different mathematical content domains as well. We elaborate on the domain specificity of professional noticing of students' mathematical thinking in the next section.

Domain Specificity: Figural-Pattern Generalisation

One assumption on which we base this study is that teachers' professional noticing expertise is domain specific (Jacobs & Empson, 2016; Nickerson, Lamb, & LaRochelle, 2017; Walkoe, 2015). We situate our teachers' professional-noticing expertise in the context of algebraic thinking, specifically, figural-pattern generalisation. We chose generalisation because of its foundational nature in the domain of algebraic reasoning (Kaput, 1998; Lee, 1996; Mason, 1996; Sfard, 1995; Smith, 2003; Stacey & MacGregor, 2001). Kaput (1998) even claimed that the act of generalising acts as a kernel from which all other forms of algebraic reasoning grow. We narrow our focus to figural-pattern generalisation (vs. numerical-pattern generalisation), which involves a sequence of figures that change in predictable ways because related tasks support students to build on and generalise their arithmetic knowledge, develop their quantitative-reasoning skills (Smith & Thompson, 2008), grapple with functional concepts, and develop meanings for algebraic symbols. Developing meanings for algebraic symbols and using them to represent situations is prominent in the U. S. secondary school standards (National Governors Association Center for Best Practices (NGA) & Council of Chief State School Officers (CCSSO), 2010). Therefore, we believe that teachers who develop proficiency in professionally noticing students' ideas in this content domain can support their students' development of a wide range of algebraic concepts.

In the next two sections we describe the strategies students employ in figural-pattern-generalisation tasks and the problems that influence students' strategies. We organise this section around these two areas because they align with professional noticing: Teachers attend to and interpret students' *strategies* and decide on *next tasks* to appropriately challenge students given their current mathematical understandings.

Students' generalisation strategies. We assume that figural-pattern tasks are designed to help students develop meanings for symbols by supporting the making of connections between symbolic expressions and quantities in the pattern. Hence, we describe strategies that involve reasoning with the figural pattern (Becker & Rivera, 2005; Healy & Hoyles, 1999; Lannin, Barker, & Townsend, 2006; Stacey & MacGregor, 2001) rather than, say, creating a formula to fit a table of input and output values (Becker & Rivera, 2005).

To illustrate these strategies, we provide examples of ways to find Stage 9 in the pattern in Figure 1.

1. Students draw pictures of successive stages and count the number of squares in Stage 9.
2. Students notice the recursive pattern and add the differences between two stages several times to reach the desired stage (e.g., Stage 4 has two more squares on the right than Stage 3, so to reach Stage 9 from Stage 4, first add two more squares for Stage 5, then two more for Stage 6, two more… and two more for Stage 9, getting 19).
3. Using the recursive pattern, students *chunk* (Lannin et al., 2006) the stage-number differences to reach the desired stage (e.g., Stage 9 is five more than Stage 4, so we add 5 x 2 to 9 to get 19 for Stage 9).
4. Students identify a functional relationship between pattern and stage number and use this relationship to create the pattern for any stage number (e.g., each figure has one square on the left and columns of two squares, with the number of columns equal to the stage number, so Stage 9 has 1 + 9x2, or 19 squares).

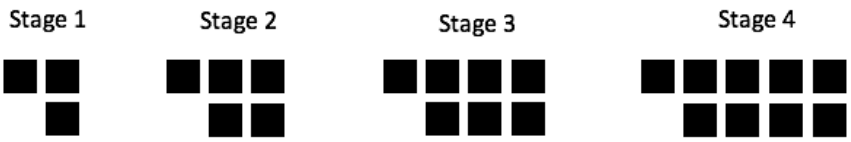

Figure 1. Example pattern in a figural-pattern-generalisation task (Jurdak & El Mouhayar, 2014).

Each of these strategies involves reasoning with the figural pattern in a certain way and, thus, elevates specific understandings of the relationships embedded in the pattern and the symbols representing these relationships (Lannin et al., 2006). For example, the second strategy is an abstraction of the first strategy that enables students to mentally visualise which objects are added next. Lannin et al. noticed that their two participants often used recursive strategies before developing other strategies to make sense of a pattern. Using the third strategy, students *chunk* several instances of the second strategy and coordinate these chunks with the difference in stage numbers. The last strategy, creating a functional relationship between the stage

number and number of objects in the pattern, usually emerges after the student has worked some time with the pattern. Functional relationships may emerge from chunking strategies, but if they do not, Lannin et al. noted that developing a strong visual image of the problem situation is an important precursor to creating a functional relationship.

Types of generalisation tasks. We distinguish figural-pattern-generalisation tasks along two dimensions. In the first, the distance of the generalisation is varied: immediate tasks, near tasks, far tasks, and finding the general rule (Jurdak & El Mouhayar, 2014; Becker & Rivera, 2005; Swafford & Langrall, 2000). Immediate tasks require the student to find the number of objects in the next stage of the pattern (e.g., Stage 4 with three stages drawn). In near tasks one finds a stage further than the next stage but close enough to solve by drawing a pattern and counting (e.g., Stage 9). For far tasks, for which drawing and counting are unproductive (e.g., Stage 100), teachers might ask students to create a function (or rule) that gives the number of objects for any stage number *n*). Immediate, near, and far tasks provide useful scaffolds for students to develop a meaningful general rule (Lannin et al., 2006).

The second dimension on which figural-pattern-generalisation tasks vary is the type of pattern exhibited (Friel & Markworth, 2009). For example, a pattern may or may not include a constant term. In Figure 1, the pattern always has one block on the left, but except for this block, the number of blocks and the figure number are directly proportional (i.e., the number of blocks is twice the stage number). Including this extra block on the left makes the pattern more difficult for students to generalise. Alternatively, one can distinguish between patterns that grow linearly (e.g., pattern in Figure 1) or nonlinearly (e.g., quadratic or exponential functions), which can be more challenging for students because the differences between successive terms change, making the chunking strategy difficult to use.

Overall, knowledge of students' strategies and of the ways figural-pattern tasks vary is an important aspect of teachers' professional noticing expertise. Understanding the connection between the two can help a teacher provide next tasks that appropriately challenge students and support their understandings. We refer to these strategies and task variations in our analysis of our teachers' responses and again in the discussion section. In the next section, we summarise the literature about teachers' professional noticing of students' mathematical thinking in the domain of algebra, a superset of figural-pattern generalisation.

Professional Noticing of Students' Algebraic Thinking

We found only two studies (Lesseig et al., 2016; Simpson & Haltiwanger, 2016) in which researchers examined secondary school teachers' professional noticing of students' thinking in the domain of algebra. Lesseig et al. (2016) created an interview module to support their prospective teachers' professional-noticing expertise of students' mathematical thinking about linear equations and analysed pre-post differences. They found that the interview module, in which teachers interviewed students regarding their mathematical thinking about linear equations, supported their teachers in attending and interpreting but not in deciding how to respond. Simpson and Haltiwanger (2016) compared the professional-noticing expertise of groups of prospective teachers on the basis of their year in the teacher education program and found that students who had more time with students than their peers exhibited more evidence of considering the students' mathematical thinking. Overall, both studies (Lesseig et al., 2016;

Simpson & Haltiwanger, 2016) provided evidence that experiences working with students support teachers' professional noticing expertise.

On the basis of the Lesseig et al. (2016) and the Simpson and Haltiwanger (2016) studies, we identified three gaps that we address in our investigation. First, in both studies, participants were prospective teachers rather than practising teachers. Thus, the influence of teaching experience, a feature that has been shown to be influential for practising elementary school teachers (Jacobs et al., 2010), was not captured in either study. Second, we focus on the content domain of figural-pattern generalisation, not a focus of either study. Finally, researchers in both studies examined differences in the degree to which teachers professionally noticed students' mathematical thinking, either within a particular group of teachers over time (Lesseig et al., 2016) or between groups of teachers (Simpson & Haltiwanger, 2016). In addition to focusing on the degree to which teachers professionally noticed students' mathematical thinking, we contribute to a characterisation of practising teachers' professional-noticing expertise, which may be used to support professional developers' knowledge base related to their teachers' understandings (Jacobs et al., 2011). In particular, we identified two research questions:

1. Prior to sustained professional development on students' mathematical thinking, to what degree do *practising* secondary school teachers in our study professionally notice students' mathematical thinking in the context of figural-pattern generalisation?
2. Prior to sustained professional development on students' mathematical thinking, what is the nature of the *practising* secondary school teachers' professional noticing expertise in the context of figural-pattern generalisation?

Methods

In the next three sections we describe our participants, our data-collection methods, and our analysis of their professional-noticing expertise.

Participants

We studied 16 mathematics teachers who were about to begin a professional development project focused on (a) improving their practices and (b) becoming leaders of their respective teaching communities. The teachers' years of experience ranged from 2–30 years, averaging 13 years. One participant had two years experience; the others had five or more years experience. All teachers came from high-needs school districts[4] in the Southwestern United States. Seven teachers taught students aged 11–14, and nine teachers taught students aged 14–18. At the time of data collection, teachers were becoming familiar with the Common Core State Standards (NGA & CCSSO, 2010) but had not begun teaching curricula to address these standards.

[4] In the No Child Left Behind Act (2002) a *high-needs school* is defined as within the top quartile of elementary and secondary schools statewide in the number of unfilled, available teacher positions or as located in an area with at least 30 percent of students from families with incomes below the poverty line, a high percentage of out-of-field-teachers, high teacher-turnover rate, or a high percentage of teachers not certified or licensed.

Because the purpose of the professional development was to support teachers to both enrich their teaching practices and become teacher leaders, we selected the top third of the applicants on the basis of their knowledge of mathematics, effective teaching practices, and positive dispositions toward learning and growing in their practices. In addition, teachers were required to be teaching either a prealgebra or an algebra course, in which figural-pattern-generalisation tasks tend to be used.

Data Collection

In this study, we were interested in our teachers' professional noticing of students' mathematical-thinking expertise (Jacobs et al., 2010). Hence, similar to Jacobs and her colleagues, we used a video-based classroom artefact that exhibited student thinking and asked teachers to respond to noticing prompts related to the artefact. Prior to watching the video, teachers solved the task so were familiar with it. After watching the 8-minute video, they completed a written assessment in response to three writing prompts, detailed further on in this section Because the data reported here were collected *prior to* the teachers' engagement with the professional development project, we hope to provide mathematics professional developers a starting point from which to build understanding of the degree and nature of practising secondary school teachers' professional noticing of students' pattern-generalisation thinking.

Beams Task video. To provide teachers opportunities to notice several features of a mathematics lesson, we sought a classroom video in which students worked together, creating multiple representations and using multiple strategies to solve a generalisation task. On the basis of these criteria, we chose the Beams Task video clip from the *Research-Based Practices for Teaching and Learning* (Carpenter & Romberg, 2004).

The video shows students engaging with a task called "Building Formulas" (National Center for Research in Mathematical Sciences Education (NCRMSE), 2003, p. 26). Students in the clip are 13–14 years of age. In the clip, the teacher presents a visual of the beams (Figure 2), and students share in small groups any patterns they notice. This pattern grows linearly because it increases by four each time and has a constant term because the number of rods is always one less than a multiple of 4 (Friel & Markworth, 2009). Next, students share with the class the patterns they found. Some students share recursive patterns and identify a recursive formula; a common strategy students use to explicate figural patterns (Jurdak & El Mouhayar, 2014). The teacher builds on the recursive formula to pose the main task: to create an explicit formula for the total number of rods needed, given any length. The video then shows two student groups working on the main task.

At the end of the video, two students, Tristian and Beverly, each share their group's solutions (Figure 3). Tristian describes her group's solution as separating the beams pattern into a top, a middle, and a bottom portion. Using L for the number of bottom rods, she notes that the top portion has one rod fewer than the bottom portion, and the middle (diagonals) portion has twice as many rods as the bottom portion. Thus, her group's formula for the sum of the three portions is (L) + (2L) + (L – 1). According to the literature, this formula is a functional rule because it describes an explicit relationship between the length *L* and the total number of rods (Lannin et al., 2006).

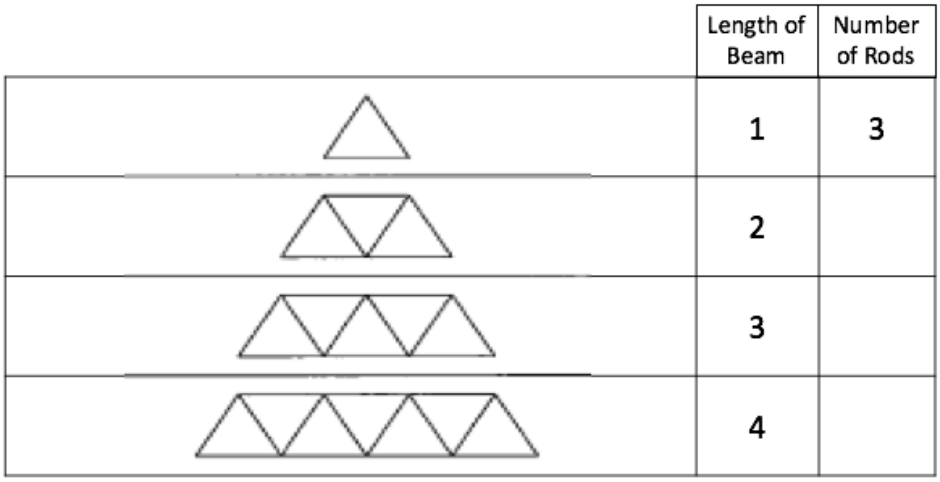

Figure 2. Beams Task (NCRMSE), 2003).

Beverly shares the formula $4L - 1$, which she shows is correct for a beam of length six. To explain the formula components, she identifies a 4-rod unit as a triangle and a horizontal beam on top with the beam as a composition of these 4-rod units, explaining multiplying L by 4. She notes that the last unit lacks a top beam, explaining the – 1 in their formula. This formula relates to two strategies in the literature: (a) a chunking strategy in repeated addition of 4-rod units in their formula and 4-rod units derived from the difference between successive terms and (b) a functional rule expressing the explicit relationship between the length L and the total number of rods (Lannin et al., 2006).

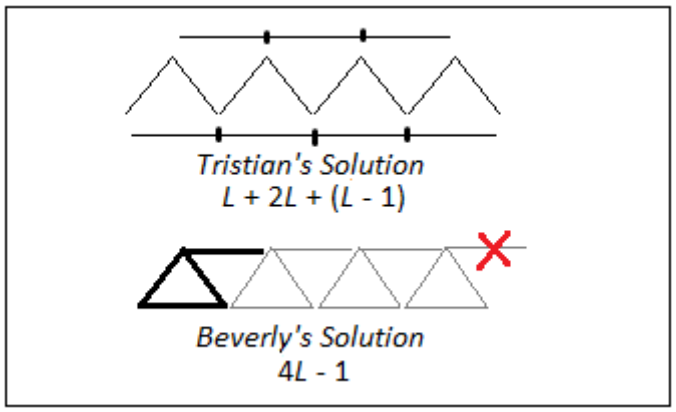

Figure 3. Tristian's and Beverly's ways of *seeing* the beams.

Writing prompts. The professional noticing of students' mathematical thinking assessment consisted of three prompts, developed by Jacobs and her colleagues (2010). Each prompt addressed a different component skill of noticing:

1. Please describe in detail what the students who shared their formulas at the white board said and did in response to this problem. (We recognise that you had the opportunity to view this video only one time, so please just do the best you can.)
2. Please explain what you learned about these students' understandings.
3. Pretend that you are the teacher of these students. What problems might you pose next? (We are interested in how you think about selecting problems, but we do not believe that there is ever a "best" problem, and we recognise that as the teacher of these students you would have more information to inform your selection.)

Time for responding to the prompts was not restricted, and, as in other professional noticing studies (Jacobs et al., 2010; Jacobs et al., 2011), teachers could not rewatch the video because professional noticing is an in-the-moment classroom practice.

Data Analysis

To reduce bias during the coding process, all responses were blinded. To answer the first research question, the first three authors followed Jacobs et al.'s (2010) coding process and scored responses indicating the evidence for participants' levels of engagement with the students' mathematical thinking as *robust evidence*, *limited evidence*, or a *lack of evidence*. Although we recognise that teachers might have more robust professional noticing expertise than a written assessment captures, we had access only to the professional noticing expertise *evidence* demonstrated in the written responses. We independently coded first the attending responses, then the interpreting responses, and finally the deciding-how-to-respond responses and then resolved any discrepancies in scores through discussions. Interrater reliability for each prompt was greater than 80%.

To answer the second research question, we grouped responses with the same level of evidence and looked for patterns within each grouping to understand the nature of each level of our teachers' professional-noticing expertise. We next describe our scoring of responses to assess the degree to which participants considered the students' thinking.

Attending responses. We followed Jacobs and her colleagues' (2010) three steps for coding the *attending* responses. First, we identified the mathematical details for each student's solution. For Tristian,

1. Her formula had three components, namely L, (L – 1), and 2L.
2. She could name the referent in the picture for each component.
3. She articulated that twice as many beams were in the middle as in the bottom.

For Beverly,

1. She decomposed the beam into a succession of 4-rod units made up of a triangle with a horizontal rod on top (a picture sufficed).
2. She subtracted 1 because the final "shape" was missing the horizontal rod.
3. She wrote, "4L – 1 = Total."
4. She showed that the formula gave the correct result for a beam of length six.

We counted the number of mathematical details the teacher attended to in each student's solution and scored each response as lack of (0 – 1 details), limited (2 details), and robust

evidence (3 – 4 details) of attention to the details in the student's solutions. For each teacher's single score, if the scores for the two students differed by two levels, we gave the average score; if the scores differed by one level, we used the higher score because they had provided evidence of attending to student thinking at that level. Note that Jacobs et al. used only two levels of attending, whereas we used three.

Interpreting responses. We scored *interpreting* responses on the basis of the extent to which the response reflected a consideration of the students' understandings and were consistent with the students' strategies (Jacobs et al., 2010). Strong indications of such interpreting included providing specific details of understandings that were consistent with the students' strategies, and differentiating understandings between students. A response in which a teacher simply redescribed the students' strategies without making inferences about the students' understandings, or made inferences inconsistent with the students' strategies, lacked evidence of the teachers' interpreting the students' understandings. Using these criteria, we scored responses using the same three levels that Jacobs et al. (2010) used: *lack of evidence, limited evidence,* and *robust evidence* of interpreting students' understandings.

Deciding-how-to-respond responses. We scored deciding-how-to-respond responses on the basis of the extent to which the teacher considered the students' mathematical understandings (Jacobs et al., 2010). Hence, our focus was on both the problem posed and the rationale provided. Rationales that included recognising how a particular problem could enable these students to build on the understandings demonstrated in the video, or anticipating how the students in the video might respond to the problem, earned higher scores than responses inconsistent with the students' understandings. Specific details, insightful comments, and individualised problems and rationales were strong indications of considering the students' understandings when deciding how to respond. A teacher's posing a task and expressing curiosity for how students might complete the new task showed openness toward learning about the students' understandings (Jacobs et al., 2011). In contrast, a response designed to force a particular strategy indicated that a teacher was not open to the students' individual understandings. Using these criteria, we scored responses using the same codes as Jacobs and her colleagues: *lack of evidence, limited evidence*, and *robust evidence* of considering the students' understandings when deciding how to respond.

Findings

We structure our findings on the basis of our two research questions. First, we share the degree to which the participants considered the students' solutions and ideas when attending, interpreting, and deciding how to respond. Then, for each component skill, we describe qualitative trends among the responses we observed and present illustrative examples of responses receiving high and low scores along with other responses that received the same score within that component-skill.

Degree to Which Teachers Considered Students' Mathematical Thinking

See Table 1 for a summary of the descriptive statistics for each component skill of professional noticing. Our teachers showed moderate evidence that they could attend to the mathematical

details of the students' strategies, with 75% providing at least some evidence of attending to the details and two thirds of that 75% demonstrating robust evidence. Hence, prior to professional development, many of our secondary school mathematics teachers were able to attend to the mathematical details of students' solutions to the figural-pattern-generalisation task.

The teachers provided less evidence of both being able to interpret the students' mathematical understandings and considering the students' mathematical understandings when deciding how to respond. In particular, none of our teachers provided robust evidence of considering the students' mathematical understandings, only half provided some evidence of interpreting the students' mathematical understandings, and only a quarter provided some evidence of considering the mathematical understandings when deciding how to respond. As we describe later, many teachers in this study focused on ideas other than the students' mathematical understandings when responding to Prompts 2 and 3.

Table 1
Degree of Professional Noticing of Secondary Teachers (Total N *= 16)*

Component skill	Lack of evidence	Limited evidence	Robust evidence
Attending	4 (25%)	4 (25%)	8 (50%)
Interpreting	8 (50%)	8 (50%)	0 (0%)
Deciding how to respond	12 (75%)	4 (25%)	0 (0%)

Attending to students' strategies

Half of our teachers provided robust evidence of attending to the mathematical details of the students' strategies, a quarter provided some evidence, and a quarter provided little evidence of attending to the details. In the following subsections, we characterise responses that demonstrated robust evidence and responses that demonstrated a lack of evidence. See Table 2 for representative examples of attending responses. The participants' names have been replaced with pseudonyms.

Table 2
Examples of Attending to the Details of Beverly's Strategy

Lack of evidence (Alana)	Robust evidence (Betty)
She also presented her formula and connect [*sic*] the symbols with the model they had built.	She stated that she viewed the problem as groups of 4 [image in Figure 4] that were joined together. She said you have a group of 4 rods for every 1 beam (the bottom piece) so you multiply 4 by the # of beams. She then said that the last group of 4 lacked the 4th rod so you subtract 1. She showed on the board what made a group of 4 [see image in Figure 4] and drew rod 4 on the last group to show where the -1 is depicted.

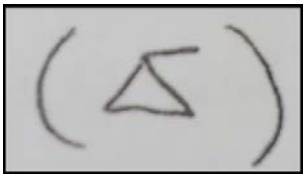

Figure 4. Betty's picture of Beverly's groups of 4.

Robust evidence. Betty's response is representative of other responses that included robust evidence of attending in that she provided specific details about Beverly's strategy and included three of the four mathematically significant details we described in the methods section. Pictures were not required for a robust-evidence score, but Betty's picture confirmed her attention to the mathematical details.

In general, responses to the attending prompt scored as robust did not vary in significant ways. All responses in this group tended to be as specific as Betty's and provided observable details of each student's response. The content of their responses did not deviate from the mathematical details we had selected.

Lack of evidence. Alana's response is representative of other responses that lacked evidence of attending to the details of Beverly's strategy. Notice that Alana included general statements such as "She … connect[ed] the symbols with the model they had built," which although true does not provide evidence that Alana attended to *how* Beverly connected the symbols to the model, important mathematical details.

The four responses that lacked evidence of teachers attending to students' ideas included other general statements. One participant wrote, "She [Tristian] was able to articulate what each part of the formula represented." Although true, this statement provides no evidence that the teacher attended to *how* Tristian explained her solution. Additionally, two teachers provided vague inferences rather than details about what the students said and did. For example, one

participant claimed, "Beverly saw the connection in a simpler way," without explaining the "simpler way."

Interpreting students' understandings

Half of our teachers provided evidence of interpreting students' mathematical understandings in ways consistent with the students' strategies, but none of these teachers provided robust evidence. Hence, we describe qualitative trends among responses that provided limited evidence or lacked evidence of interpreting the students' mathematical understandings. See Table 3 for representative examples of interpreting responses.

Limited evidence. Betty's response provided limited evidence of interpreting the students' understandings. Notice that Betty's claims were mostly consistent with the students' solutions and that she described understandings rather than simply redescribing the students' solutions. For example, the claims "these students understand properties of operations" and "these students use[d] pictures to check if a pattern always works" were both understandings consistent with the students' strategies, and both were beyond the students' observable actions.

However, Betty's response demonstrated limited evidence, *not* robust evidence, of interpreting the students' understandings because her claims were vague, and she tended to overattribute understandings to the students in writing that the students understood "the definition of 'variable,'… [and] independent and dependent variables," although the video lacked evidence for either claim. If Betty had written instead that "the students in the video were comfortable using a letter to represent a varying quantity and then using that symbol in an expression to capture its relationship to other quantities," the response would have been specific and consistent with the students' strategies, providing robust evidence that the teacher interpreted understandings on the basis of the students' strategies.

Table 3
Examples of Interpreting *Responses*

Lack of evidence (Carina)	Limited evidence (Betty)
I can see that students can solve a problem given the correct amount of time and resources. I really appreciated how they can discuss and collaborate with one another on how to solve the problem. I hope to be able to do more structured tasks like this, especially in the linear functions unit. Sometimes, we teachers are so crunched for time that we simply "tell" the students the information they need to know whereas if we allow them the time to explore, they can come up with not only multiple ways to solve a problem, but are also able to comprehend & remember the content well; and be able to translate this knowledge to other parts of the curriculum.	These students (especially those that presented) generally understand properties of operations, generalising patterns, making tables to track data, use of variables and the definition of "variable," use pictures (manipulatives) to check if a pattern always works, [and] independent and dependent variables. I do wonder about Tristian's group and if they could see the like terms in their pattern. Could the teacher have pushed this thinking?

In the eight responses that provided limited evidence for interpreting the students' understandings, participants tended to make correct but vague statements about the students' understandings, with some overattributions. Specifically, many teachers commented that students could make connections between the formulas and the visual pattern, such as "They were also able to understand how a formula could be used to describe a pattern" (n = 6). Teachers also said that the students "understood variables," or "could use a variable in a formula" (n = 5). Four teachers commented that students saw the patterns in different ways, and three commented that students could verbally describe the patterns they saw.

Lack of evidence. Carina's response provided no evidence that her interpretation of either student's mathematical understandings was based upon consideration of the students' understandings. Notice that Carina focused on other aspects of the classroom: the students' discussing and collaborating to solve the problem. In particular, Carina commented on general pedagogical issues with which she grapples in her instruction rather than on the students' mathematical understandings evident in the video, and so her response was scored as providing a lack of evidence of interpreting the students' mathematical understandings. That said, Carina's orientation to the video (for example, her appreciation for the students' abilities to collaborate and her desire to use tasks like the one shown in the video) may provide a starting point for professional developers; we address this point in the discussion.

In the eight responses that lacked evidence of interpreting students' understandings, three were focused on general pedagogy and described other aspects of the class that intrigued them rather than on the students' mathematical thinking. For example, one teacher commented, "I learned that it is incredibly difficult to really know how our students think through a problem when all we do is evaluate their written work." Another trend was to make claims about the students' understandings for which no evidence appeared in the video (n = 3). For example, one

teacher claimed, "Beverly… would do well with fractions because she was able to visualise a unique whole." Teachers also tended to redescribe what Beverly and Tristian did at the board (*n* = 3). Finally, four teachers' claims were too vague to provide any evidence of interpreting the students' understandings; one response, for example, included the claim that "Beverly's approach was global."

Deciding how to respond on the basis of students' understandings

A quarter of our teachers provided evidence that they were deciding how to respond to these students on the basis of the students' understandings, and three quarters did not. As for the interpreting responses, no response in this component skill demonstrated robust evidence of considering the students' mathematical understandings. Hence, for the following subsections we describe qualitative trends by comparing responses that provided limited evidence with those that lacked evidence of considering the students' mathematical understandings when deciding how to respond. See Table 4 for representative examples of deciding-how-to-respond responses.

Table 4.
Examples of Deciding-How-to-Respond Responses

Lack of evidence (Ellen)	Limited evidence (Felisha)
Problem: Any problem that demonstrates a linear growth pattern incorporating a different rate (either decreasing (m = -3) or other increasing rate that is rational (m = 3/2)). Rationale: Students can gain a sharp sense of linear relationships when given opportunity to discover constant rates of increase and decrease in one quantity with respect to another. They can then apply this to slopes of lines when working with abstract functions.	Problem: Given this pattern [image in Figure 5] draw the next two figures and write the equation for the pattern if x represents the figure number and y represents the number of tiles. Rationale: I'd give them [this] problem to see how they would apply their knowledge from the first problem in answering the next one. I'd also be interested in seeing if students would connect it to the graph on their own, or create one.

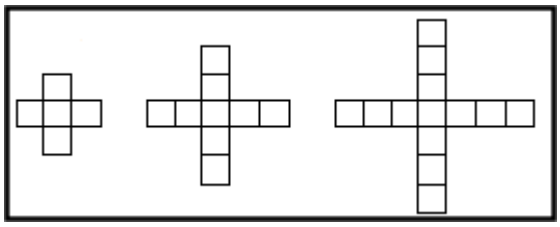

Figure 5. Pattern posed by Felisha.

Limited evidence. Felisha's response is one of the four responses that provided limited evidence for deciding how to respond on the basis of the students' understandings. Notice that Felisha focused her rationale on seeing *how* these students could use their knowledge to solve a similar problem, indicating that Felisha held an orientation toward having students use their own ways of reasoning and building on these ways of reasoning. Consistent with the coding scheme of Jacobs et al. (2011), expressing an interest in understanding and providing space for students' mathematical ideas indicated that the teacher considered the students' mathematical understandings when deciding how to respond.

However, Felisha's response was not coded as robust because wondering how Beverly or Tristian would apply their knowledge to the next task is a vague inquiry. In addition, the inquiry about graphs, in particular, appears to reflect the teacher's curricular goals rather than a decision based on the students' understandings. Alternatively, if Felisha had wondered if Beverly's group would respond to this new problem by looking at what is added on from one stage to the next and using this difference to generalise the pattern, consistent with what Beverly's group did in the beams task, her response would have provided robust evidence instead of limited evidence that she decided how to respond on the basis of the students' understandings.

Four participants demonstrated limited evidence of considering the students' understandings when deciding how to respond: Felisha posed a new generalisation task, two others asked students to compare the two formulas presented, and one suggested giving students a new length to consider. All four teachers demonstrated limited evidence because they built on Beverly's and Tristian's thinking, or they expressed a curiosity and openness for new ways of reasoning. For example, one who posed a comparison problem shared the following question in the rationale: "Do the students in the other groups understand how each group came to the formula?" This question provided evidence that the teacher was curious about what other students understood from Beverly's and Tristian's presentation and thus demonstrated some evidence of considering the students' mathematical thinking.

Lack of evidence. Ellen's response is representative of other responses that lacked evidence for deciding how to respond on the basis of the students' understandings. Notice that Ellen did not connect her rationale to Beverly's or Tristian's solution to the problem but instead mentioned "students" in general, and outlined a progression of mathematical topics to present without explaining why these topics were appropriate next steps for these students. Ellen's decision seems to have been driven by mathematical goals that did not necessarily relate to what she saw Beverly or Tristian do. We acknowledge that this progression of topics or problems may be appropriate for some students; however, this response did not provide evidence that the teacher considered Beverly's or Tristian's mathematical understandings when deciding how to respond.

Of the 12 participants who provided a *lack of evidence* for deciding how to respond on the basis of the student's understandings, eight posed new patterns for students to generalise. Of these eight, five participants posed a new pattern to repeat the generalisation process (e.g., "This problem is generalizable, so they can be scaffolded to come up with a formula like the first problem"). Although we agree with these teachers that practice is an important part of learning, we saw no evidence that these teachers considered the students' generalisation strategies other than knowing that students should practise what they learn. The other three participants who

posed new generalisation patterns proposed introducing new mathematical concepts (e.g., *y*-intercepts and slope). Again, we recognise that these may be logical directions for teachers to pursue, but we saw no evidence that these teachers connected their rationales to the mathematical thinking of Beverly and Tristian, except possibly for the fact that Beverly and Tristian wrote an explicit formula.

Another large majority of responses that lacked evidence included comparisons of the two formulas presented (n = 4) or creation of a graph to represent the functional relationship (n = 3). As a reminder, we focused on the rationales behind the problems posed to identify evidence of teachers' considering the students' understandings, and all seven of the tasks posed were focused on new mathematical content without explicitly linking this content to the students' mathematical understandings. For example, one teacher asked students to compare Tristian's and Beverly's formulas because doing so "teaches combining like terms." Another teacher asked students to graph the relationship because "graphing is important." Suggesting new mathematical content without relating it to Beverly's or Tristian's thinking in the rationale did not demonstrate evidence of considering the students' mathematical understandings. The last two respondents suggested extending the Beams Task to include "hourly labour costs" and checking formulas for a beam of length seven.

Across the 16 responses to this prompt, we found six distinct categories of problems that teachers posed: (a) prompts to generalise a new pattern (n = 9), (b) prompts to compare Beverly's and Tristian's solutions (n = 6), (c) prompts to make a graph that represents the functional relationship (n = 4), (d) one prompt to reason about a beam with a new length, (e) one prompt for checking work, and (f) one extension of the problem to include labour costs and costs of the beam. Because some participants posed two problems, the numbers sum to 21. Like Jacobs et al. (2010), we scored the response based on the problem/rationale that would yield a higher score when more than one problem/rationale was provided.

Discussion

Elementary school teachers who experience sustained professional development focused on students' mathematical thinking develop more sophisticated professional noticing skills than teachers who lack such experience (Jacobs et al., 2010), but little research exists regarding these skills for practicing secondary school teachers. In this study we investigated the professional noticing skills of 16 secondary school mathematics teachers who were selected on the basis of their teaching effectiveness. We situated the study of these teachers' professional noticing in the context of figural-pattern generalisation. To summarise, three fourths of the participants provided either limited or robust evidence that they could attend to the details of students' strategies. One half of the participants provided limited evidence, and none provided robust evidence that they could interpret students' understandings. One fourth of the teachers provided limited evidence (and none robust) that they could decide how to respond on the basis of students' understanding. We discuss these findings in the context of the literature on figural-pattern generalisation and propose potential resources that professional developers might draw upon in their work with teachers.

Teachers' Responses as Resources

In this section we share our view of teachers' responses as resources upon which professional developers can draw when supporting teachers to develop their professional noticing of students' mathematical-thinking expertise (Hammer, 1996; Smith, diSessa, & Roschelle, 1993; Whitacre & Nickerson, 2016). We do so by relating the content of their responses to each prompt to the literature on students' mathematical thinking in figural-pattern generalisation (e.g., Jurdak & El Mouhayar, 2014; Lannin et al., 2006; Rivera & Becker, 2008; Stacey & MacGregor, 2001). We address first the attending component skill, then interpreting, and finally deciding how to respond.

Attending to students' mathematical thinking. Our findings in regard to our teachers' attending skills seem to indicate that many of these teachers may not need significant support in this component skill, although still half of the teachers did not describe the mathematical details of the students' solutions. Because attending is most likely foundational for interpreting and deciding how to respond (Jacobs et al., 2010), we entreat professional developers to not overlook this component skill. As Jacobs et al. pointed out, attending "requires not only the ability to focus on important features in a complex environment but also knowledge of what is mathematically significant, and skill in finding those mathematically significant indicators" (p. 194). In our case, knowledge that supports teachers' attending skills might include knowledge of students' strategies for solving these tasks and knowledge that students visually decompose patterns in different ways (Healy & Hoyles, 1999; Lobato, Ellis, & Munoz, 2003). For example, Swafford and Langrall (2000) provided evidence that different ways of counting a pattern led to different symbolic generalisations. With such knowledge, teachers might become predisposed to attend to plausible ways a student counted to create an expression rather than solely attending to the expression. Professional developers might support teachers' attending skills by discussing research about the types of strategies students use when solving these tasks.

Interpreting students' mathematical thinking. Overall, teachers interpreted many mathematical understandings from the students' strategies. One such understanding was that the students saw the patterns in different ways, a notion that researchers have identified (Bishop, 2000; Swafford & Langrall, 2000). Professional developers could build on such teachers' noticings in attending to further explore implications for the multiple ways of seeing the pattern and thus explore the specifics of students' generalisation strategies and their relationship to students' understandings (Lannin et al., 2006). For example, Lobato, Hohensee, and Rhodehamel, (2013) shared that when students in one classroom focused on the growth of a pattern (i.e., the numeric difference between one figure and the next), many students lost track of the relationship between the figure number and the total number of objects in the pattern. By attending to the details of strategies focused on additive growth, teachers can interpret these strategies as evidence of recursive thinking. Providing teachers with explicit opportunities to reflect on students' strategies and how the strategies may influence students' understandings should be a hallmark of the professional development.

The teachers also noticed that students made connections between their symbols and the representation and that students verbalised these connections (Bishop, 2000; Ellis, 2007; Stacey & MacGregor, 2001). Teachers can begin to see verbalisation in students' solutions, an important idea that deserves further investigation by professional developers and teachers. In

fact, Stacey and MacGregor (2001) found evidence that students who verbally described the patterns they saw created an explicit symbolic rule significantly more often than students who did not. Professional developers might draw upon such teachers' noticings to discuss how students develop meanings for symbols, and the role of verbalisation.

Another category of responses that provided a lack of evidence of interpreting students' understandings also emerged as a resource. Many teachers responded similarly to Carina, who expressed a desire to use tasks similar to those she saw in the video and who appreciated students' abilities to collaborate. Although such responses lack evidence of interpreting students' understandings, they do provide evidence of the appreciation of student collaborations and motivation to use the type of task shared in the video. These responses provide professional developers with the knowledge that at least some teachers value having students collaborate and desire to change their practices. Professional developers can use these common goals and values as starting points for engaging deeply with students' mathematical understandings.

Overall, our teachers noticed several mathematical understandings in the students' solutions but still made general claims when interpreting students' understandings, tending to overattribute the students' mathematical understandings, a feature seen in prospective teachers' professional noticing expertise as well (Simpson & Haltiwanger, 2016). Hence, professional developers might support teachers to develop more nuanced knowledge bases related to the students' ways of reasoning they notice in the students' solutions. As a last example, noticing that the students understood variables is vague and oversimplified, but it could be a starting point for discussing the many meanings of *variables,* which are much more complex than educators might first expect (Chazan & Yerushalmy, 2003; Philipp, 1992).

Deciding how to respond on the basis of students' mathematical understandings. To identify teacher resources in this component-skill, we focus on the types of problems teachers posed. We believe that many of the problems posed by the teachers could be productive for supporting student learning, but because the associated rationales were focused on introducing new mathematical concepts (e.g., rate, slope, y-intercept, graphing, and combining like terms), they lacked evidence of the teachers' considering the students' mathematical understandings. Similarly, Glassmeyer and Edwards (2015) found that teachers initially often fail to value generalising and making sense of algebraic notation, important foci of figural-pattern-generalisation activities. We conjecture that prior to professional development many teachers believe that they need to address other mathematical topics rather than focus on the act of generalising, a fundamental form of algebraic reasoning (Kaput, 1998).

We believe that many of the problems teachers posed are productive starts for professional development. First, to highlight important differences between tasks such as linear and nonlinear and near- versus far-generalisation tasks (Jurdak & El Mouhayar, 2014), professional developers might build on the responses in which a new generalisation task is posed. We noticed that eight participants posed linear patterns with constant terms, and one posed a nonlinear pattern (Friel & Markworth, 2009). We wondered how many of the teachers recognised the affordances and constraints of these types of patterns.

Additionally, we noticed that only one participant who posed a new pattern created a near task and a far task (Jurdak & El Mouhayar, 2014). All other participants who posed a new pattern provided a prompt only to find a rule for the pattern. We wondered how many teachers

recognised the usefulness of different kinds of tasks in scaffolding students' mathematical thinking. Professional developers might aim to help teachers explore the affordances of choosing particular numbers as a means for supporting this thinking.

Many teachers suggested that students compare two solutions, an inclination professional developers might use to support exploring differences between the two. In particular, one might expand teachers' focus on comparing the symbolic expressions to include comparing the ways Tristian and Beverly reasoned about the figural pattern, which is often neglected (Friel & Markworth, 2009). Just as our teachers seemed to hold the goal of correcting Tristian's failure to combine like terms, which might have been considered a limitation of Tristian's group's thinking, Lesseig et al. (2016) found that many of their prospective secondary school teachers focused on responding to the limitations they noticed in the students' understandings. Professional developers might support deeper explorations by pressing teachers to move beyond these limitations and explore what understandings these students *are* exhibiting (rather than those they *are not* exhibiting), to discuss the underlying mathematical concepts and connections between solutions.

Next Steps

In this study we developed an image of secondary school in-service teachers' professional noticing of students' mathematical-thinking expertise in the context of figural-pattern generalisation. To conclude, we offer several routes for future research.

First, we focused our study on figural-pattern generalisation. Other researchers might investigate secondary school practising teachers' professional noticing of students' mathematical thinking in other important mathematical domains. These studies could provide researchers and mathematics professional developers with starting points for supporting teachers in those domains.

Second, we recognise that the artefact we selected shows two students who were able to construct correct, explicit formulas representing the relationship between beams and rods. These solution strategies are relatively sophisticated when compared with other strategies documented in the literature (e.g., Lannin et al., 2006). Even though deciding how to respond to *sophisticated* strategies is an important part of a teacher's practice (Jacobs & Ambrose, 2008), elsewhere (Nickerson et al., 2017) we argued that the field of teacher noticing would benefit from collecting more artefacts of both sophisticated *and naive* student thinking at the secondary school level.

Third, we offered ideas and directions for mathematics professional developers to take up when supporting secondary school practising teachers to develop their professional noticing expertise in the domain of figural-pattern generalisation. Researchers could explore these directions with teachers and investigate their effectiveness in supporting this expertise. In addition to our suggestions, one might take other directions to supporting teachers as seen in the literature. For example, Walkoe (2015) found that providing prospective secondary school teachers with a framework for students' algebraic thinking supported her teachers in developing their noticing expertise, whereas Lesseig et al. (2016) found that conducting interviews supported their teachers' expertise. Callejo and Zapatera (2016) found that collaboratively working on a mathematical task and exploring different solutions to the task supported their

teachers' noticing expertise. We suspect that the same would be true for practising secondary school teachers and recommend further research in this area.

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