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The Interplay Between the Beliefs and the Knowledge of Mathematics Teachers

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There is considerable research on teachers' beliefs and teachers' knowledge, but little empirical evidence exists as to the interplay between them. This article reports a study of 356 Norwegian teachers who answered a questionnaire and a multiple-choice test. Based on this study, the connections between two knowledge constructs and two beliefs constructs are examined. The findings demonstrate how different emphases on rules and reasoning are connected to different aspects of mathematical knowledge.

Literature Review

Beliefs

There is not one agreed definition of beliefs in mathematics education. Phillip (2007) is one of several authors who have provided an overview of the definitions related to beliefs and affect. He defines beliefs as "psychologically held understandings, premises, or propositions about the world that are thought to be true" (p. 259). Beliefs are a part of the affective domain together with emotions and attitudes, and are generally considered to be more cognitive than other aspects of the affective domain. Beliefs are also closely related to knowledge, as knowledge is defined as "beliefs held with certainty or justified true belief" (p. 259). Wilson and Cooney (2002) hold the view that believing is a weaker condition than knowing. In the words of Leatham (2006):

> Of all things we believe, there are some things we 'just believe' and other things we 'more than believe – we know'. Those things we 'more than believe' we refer to as knowledge and those things we 'just believe' we refer to as beliefs (p. 92).

Furinghetti and Pekhonen (2002) explored the multitude of definitions when they asked eighteen mathematics educators to state their agreement or disagreement with nine different definitions of beliefs, and also to give their own characterization of beliefs. Both the nine different definitions and the answers from the mathematics educators illustrate the differences in definitions, and the authors state that it is unlikely that a complete agreement will be reached. As a result of this investigation they recommend considering two types of knowledge; objective and subjective. Objective knowledge is the knowledge that is accepted by the mathematical community, and individuals have access to this knowledge and "construct their own conceptions of mathematical concepts and procedures, i.e. they construct some pieces of their subjective knowledge" (p. 53). Beliefs are connected to subjective knowledge. Furinghetti and Pekhonen (2002) define beliefs as connected to subjective knowledge that clarifies both the close relationship and the difference between the two, without using judgments of truth.

Beliefs are assumed to act as filters through which one sees the world (Pajares, 1992). A result is that teachers' beliefs are thought to have an impact on their practice (Wilson & Cooney, 2002). In this study, the interplay between teachers' beliefs and their mathematical knowledge is investigated. For example, some teachers might state that reasoning and argumentation are the most important aspects of mathematics, and a result may be that their focus strengthens their own learning of reasoning and argumentation in mathematics because they spend more time on these aspects. On the other hand, this view may result in less focus on the repeated practice needed for the teacher to have fluency in methods and rules in mathematics. Other teachers may believe that procedural fluency is the most important aspect of students' mathematical knowledge. These teachers will probably more often choose tasks where they use their procedural knowledge, thus getting the repeated practice needed to acquire procedural fluency. However, this belief may also lead to less time spent on reasoning and argumentation and consequently less learning about these aspects of mathematics. A strong emphasis on one aspect of mathematical knowledge can help teachers learn more about this aspect, but it can also hinder the learning of other aspects.

Taking this reasoning one step further, beliefs can present barriers and serve as affordances (Goldin, Rösken, & Törner, 2009). Beliefs can, for example, act as barriers against influence from external factors, such as curriculum changes or education. Thus, beliefs can preserve the teaching even if the curriculum and the use of mathematics in society change. Beliefs can also serve as affordances; for example, the belief that reasoning and argumentation are the most important aspects of mathematical knowledge may lead the teacher more often to situations where he learns about students' mathematical thinking.

The knowledge of how beliefs can serve as affordances is very useful in the development of teachers' productive disposition–defined by Kilpatrick, Swafford and Findell (2001) as one of five components of 'proficient teaching of mathematics'. On the other hand, when beliefs present barriers against learning or development, they need to be challenged. Beliefs are seen as more difficult to change than emotions and attitudes (Philipp, 2007). Whether or not the beliefs are available for change also depends on how they are held. Beliefs can be held without regard to evidence (non-evidentially) or based on evidence or reason (evidentially) (Green, 1998). If a belief is held non-evidentially, "it cannot be modified by introducing evidence or reason" (p. 48). When a belief is held evidentially, the individual will respect other views as reasonable and intelligent. In such cases the beliefs are available for discussion, and can be modified by further evidence or better reason. Reflection is regarded as a critical factor for changing beliefs, as "teachers learn new ways to make sense of what they observe" (Philipp, 2007, p. 281).

Beliefs can also be held with different degrees of psychological strength (centrally or peripherally) and as primate or derivate beliefs (Cooney, Shealy, &

Arvold, 1998; Green, 1998). In a belief system, some beliefs are held more strongly than others. The centrally held beliefs are harder to change, while the peripherally held beliefs are more open to discussion, examination and change. The primary beliefs are the beliefs that are so basic that they are not derived from any other beliefs, and the derivative beliefs are the beliefs that are based on other beliefs. Green (1998) refers to the structure of primary and derivative beliefs as a quasi-logical structure because this has little to do with objective, logical relations between beliefs. A belief can at the same time be derivative and psychologically central, or primary and psychologically peripheral.

Beliefs influence the decisions that individuals make and also serve as the best indicators of their decisions (Goldin et al., 2009). As a result of such a view, a lot of research has been focused on the connection between beliefs and teaching practice. However, inconsistencies are often documented between the teachers' practice and their beliefs about mathematics teaching. Inconsistent beliefs can be held simultaneously without becoming evident because they are connected to different contexts, certainty and consciousness (Törner, 2002). This raises the question of how important beliefs are for the teachers' practice.

Another perspective is to look at teachers' beliefs as sensible systems, where apparent inconsistencies are instead thought to be caused by other beliefs ranking higher in certain situations (Leatham, 2006; Skott, 2009). These beliefs might be non-mathematical, for example related to a teacher's wish to help the students succeed. Sometimes a teacher's behaviour looks inconsistent with the teacher's beliefs because the researcher is not aware of all the beliefs at play during the observed practice. When one looks at teachers' beliefs as sensible systems, such inconsistencies are investigated in order to understand which unknown beliefs decide this behaviour. Even if a belief is held as primary, it may not be central (psychologically strong), and vice versa. An example of this is when a teacher has a primary belief that practical mathematics is important to help the students understand mathematics. At the same time, it is possible that the belief lacks psychological strength. If so, the belief may not influence decisions in competition with other beliefs that are psychologically stronger. Studying the strength of beliefs and which beliefs are central and peripheral in a given situation is important in order to understand the decisions that a teacher makes.

Beliefs are connected to an object, and this object may vary from being abstract to being more concrete (Törner, 2002). It is possible to differentiate between domain-specific and global beliefs. The global beliefs about mathematics "describe very general beliefs including beliefs on the teaching or learning of mathematics, on the nature of mathematics, and on the origin and development of mathematical knowledge" (Törner, 2002, p. 86). The domain-specific beliefs are connected to the differing characteristics that different fields of mathematics possess (Törner, 2002). A procedure or a concept can be the object of a belief. The nature of the connections between global and domain-specific beliefs is still an open question. It is possible that the global beliefs influence the domain-specific beliefs strongly. On the other hand, mathematics is normally taught in separate fields and therefore it is possible that the beliefs are formed as domain-specific beliefs that are later connected to each other and generalized into global beliefs.

Mathematical Knowledge for Teaching

There are many frameworks available to help establish one or more constructs of mathematical knowledge (for example Brekke, 1995; Kilpatrick, et al., 2001; NCTM, 2000; Niss & Højgaard Jensen, 2002). However, even though there is agreement that mathematical knowledge is a prerequisite for being able to teach mathematics, there is also a growing understanding that this is not enough (Ball, 2002; Ball & Bass, 2003; Ball, Hill, & Bass, 2005; Niss, 2007). As a response, several models of mathematical knowledge needed to teach mathematics have been developed, such as the 'content knowledge for teaching' (Ball, Thames, & Phelps, 2008), 'mathematical teacher competency' (Niss & Højgaard Jensen, 2002), 'proficient teaching of mathematics' (Kilpatrick et al., 2001), 'knowledge of teaching mathematics' (Tatto, Schwille, Senk, Ingvarson, Peck, & Rowley, 2008) and the 'knowledge quartet' (Rowland & Turner, 2009).

Not surprisingly, research has confirmed that teachers need to know the mathematics they are teaching (Askew, 2008). More surprisingly, research has also shown that there is no clear relationship between the teachers' formal mathematical education and their students' learning of mathematics (Askew, 2008; Ball, Lubienski, & Mewborn, 2001; Perrin-Glorian, Deblois, & Robert, 2008). The reason that such a relationship has not been found may be that measuring teachers' mathematical knowledge just in terms of their level of formal education is not precise enough because there are probably aspects of such knowledge that are more important than others.

One suggestion in the search for aspects of teachers' mathematical knowledge that matter for students' learning is to use Shulman's (1986) distinction between subject matter knowledge and pedagogical content knowledge. One recent development resulting from this suggestion is the framework 'content knowledge for teaching' (Ball et al., 2008). This framework divides both subject matter knowledge and pedagogical content knowledge into three parts, as illustrated by Figure 1 (Ball et al., 2008, p. 403).

Specialised Content Knowledge [SCK] is defined as the "mathematical knowledge not typically needed for purposes other than teaching" (Ball et al., 2008, p. 400). One example of such knowledge is the ability to decompress or unpack mathematical methods and rules; another is the ability to assess the mathematical validity of students' suggestions or nonstandard solutions. Research has demonstrated that this aspect of mathematical knowledge has an effect on students' learning (Hill, Rowan, & Ball, 2005). Common Content Knowledge [CCK] is defined as the mathematical knowledge that is common to people who know mathematics, and not unique to teachers. It is the knowledge needed to solve mathematics problems and use terms and notation correctly. "In short, they must be able to do the work that they assign their students" (Ball et al., 2008, p. 399). There is a general agreement that a lack of this type of knowledge is associated with less successful teaching (e.g. Askew, 2008).

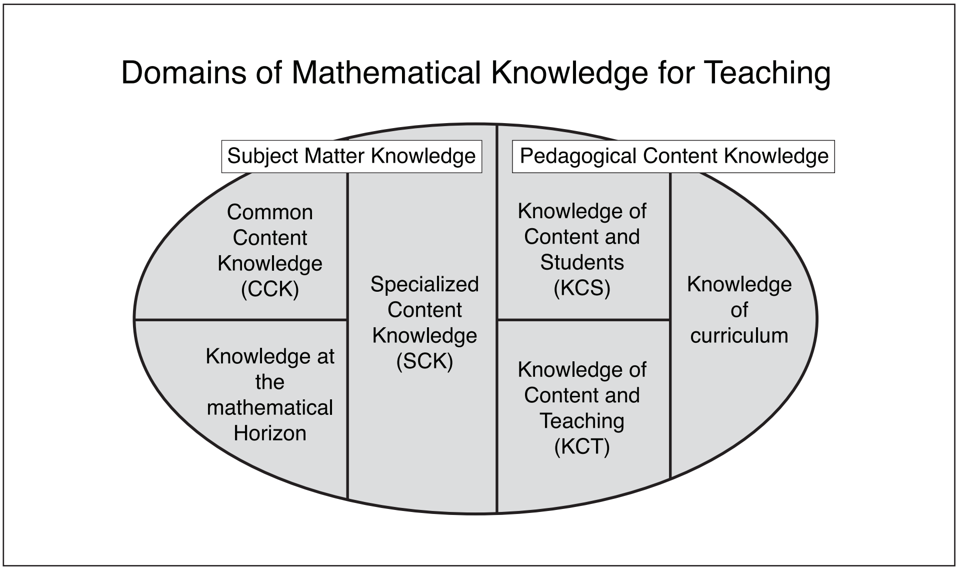


Figure 1. Content knowledge for teaching (Ball et al., 2008, p. 403)

Research Questions

It is important to find out more about how beliefs and mathematical knowledge are connected, both for pre-service and in-service teachers. This study is an investigation of in-service teachers, which provides the opportunity to include their teaching experience into the study. The main focus is on exploring the interplay between their mathematical knowledge and both their mathematical teaching experience and their mathematical education. Are these connections dependent on different types of beliefs? If so, what can the differences explain?

Method

Background

As a part of the teacher development project 'Mathematics in Northern Norway', 356 teachers of grades 1 to 10 completed an individual test and a questionnaire between October 2007 and February 2008. The teachers were from 28 schools of different sizes from both central and rural parts of the county. All teachers who taught (or normally teach) mathematics in these schools participated in the development courses of the project. Norway employs general teachers in primary and lower secondary school, and this means that almost all teachers teaching grades 1 to 7 are teachers of mathematics and thus participated in the study. For grades 8 to 10 each teacher teaches fewer subjects. In these grades, approximately one third of the teachers are teachers of mathematics and thus participated in the project.

The course leader provided standardized information about anonymity and the use of the results before the test. Some teachers worried that the management of their school would get access to the results, but they were assured that only the researchers involved in the project would have access. The course leader answered teachers' questions as long as they did not involve assistance to answer any of the tasks. No time limit was given, and nobody withdrew from the test. All together 90% of the participating teachers were tested, since 36 teachers were absent, mostly due to illness.

The development courses were mainly held outside the schools and the course leaders were all teachers in mathematics from different teacher education institutions. Every teacher participated in at least one course a year. Each course lasted for three days with about one month between each day. The teachers were given tasks to do between each course day. The courses had a variety of content focusing on mathematical knowledge for teaching (for example one course involved numbers, counting and early calculations).

The Constructs

To be able to answer the research questions it is necessary to establish one or more constructs that represent relevant mathematical knowledge, and one or more constructs that represent mathematical beliefs.

Mathematical knowledge. The test consisted of 20 tasks with a total of 46 questions originally developed by the *Learning Mathematics for Teaching* project (LMT, 2009). The test results verified the existence of the constructs of specialized content knowledge (SCK) and common content knowledge (CCK). SCK items have a reliability of 0.77 (measured by Cronbach's alpha, see Crocker & Algina, 1986) with 27 items (questions). The SCK construct consists of tasks where the considerations needed are purely mathematical, but of a type that teachers often meet in the classroom and others hardly ever will meet (see Figure 2 for an example). The SCK tasks cover different mathematical topics, and include items where the teachers should assess rules of thumbs, assess if suggestions are valid, if an invented rule works for all numbers, which text is connected to a number calculation and which answer(s) on a student task can be a good evidence that the student understands.

The CCK construct items have a reliability of 0.72 with 12 items, and consists of tasks that need purely mathematical considerations that are common to people who know mathematics (see Figure 3 for an example). The CCK tasks include decomposing of whole numbers, equivalent expressions (fractions, decimal, percent) and assessing the number of solutions to three different number tasks. The observed correlation between the SCK and the CCK construct is 0.58 and the latent correlation is 0.80. This confirms that there are two different constructs measured since the constructs turn out to be sufficiently different empirically. For further details on the analysis, see Drageset (2009).

Imagine that you are working with your class on multiplying large numbers. Among your students' papers, you notice that some have displayed their work in the following ways:

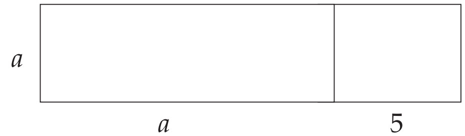
Student A	Student B	Student C
35	35	325
x25	x25	x5
125	175	25
+75	+700	150
		1000
		+60
875	875	875

Which of these students would you judge to be using a method that could be used to multiply any two whole numbers? (Tick the appropriate column.)

	Would work for all whole numbers	Would NOT work for all whole numbers	I'm not sure
Method A			
Method B			
Method C			

Figure 2. An example of a SCK task (from Hill et al., 2004)

Ms. Whitley was surprised when her students wrote many different expressions to represent the area of the figure below. She wanted to make sure that she did not mark as incorrect any that were actually right. For each of the following expressions, decide whether the expression correctly represents or does not correctly represent the area of the figure. (Tick the appropriate column.)



	Correctly represents	Does not correctly represent	Not sure
$a^2 + 5$			
$(a + 5)^2$			
$a^2 + 5a$			
$(a + 5)a$			
$2a + 5$			
$4a + 10$			

Figure 3. An example of a CCK task (from Hill et al., 2004)

Beliefs about mathematics. The first beliefs construct is based on a view that instrumental understanding (Skemp, 1976) of mathematics is the most important aspect of mathematical knowledge. The construct consists of ten statements that were answered along a four-point Likert scale (reliability of 0.71). Those agreeing with these statements emphasise formal mathematics and the learning of rules as most important, without focusing on explanations or connections. Their focus is on solving the tasks, and they do not emphasise connections or explanations. This beliefs construct is called 'rules' (see Table 1), and has clear similarities with a construct that Nisbet and Warren (2000) call 'a static view of mathematics'.

Table 1
Statements for the rules construct

Please indicate the extent to which you agree with the statements below (disagree entirely, disagree somewhat, agree somewhat, agree entirely)	
A1	The most important aspect of mathematics is to know the rules and to be able to follow them
A2	Mathematics means finding the correct answer to a problem
A4	The best way to learn mathematics is to see an example of the correct method for solution, either on the blackboard or in the textbook, and then to try to do the same yourself
A5	If you cram and practice enough, you will get good at mathematics
A6	Those who get the right answer have understood
A8	Mathematics should be learned as a set of algorithms and rules that cover all possibilities
A10	What you are able to do you also understand
AD3	In mathematics, it is more important to understand why a method works than to learn rules by heart [opposite]
Please indicate how important you think each element below is (not very important, somewhat important, important, very important)	
A11	Learning rules and methods by heart
A12	Learning formal aspects of mathematics (e.g. the correct way to write out calculations) as early as possible

The teachers that emphasise rules consider rules and the correct answer to be the most important aspects of mathematics, and feel that mathematics is best learned by rote and by trying to imitate examples. The rationale behind this belief can be that the teacher wants to reduce the complexity and ambiguity of mathematics. Another reason can be that the teacher avoids taking risks because of uncertainty in his own mathematical knowledge.

The other beliefs construct consists of eleven statements along a four-point Likert scale (reliability of 0.81). This construct is based on a view that reasoning competency (and to some extent problem solving competency) (as defined by

Niss & Højgaard Jensen, 2002) are important aspects of mathematical knowledge. The construct represents a belief that reasoning, argumentation and justification are more important than the answer. This beliefs construct is called ‘reasoning’ (see Table 2), and is connected to a dynamic view of mathematics. A focus on reasoning, argumentation and justification in practice would involve a risk of not being able to follow the students’ thinking.

Table 2
Statements for reasoning construct

Please indicate the extent to which you agree with the statements below (disagree entirely, disagree somewhat, agree somewhat, agree entirely)	
C1	The pupils learn more mathematics from problems that do not have a given procedure for solution, where instead they have to try out solutions and evaluate answers and procedures as they go
C2	It is important to be able to argue for why the answer is correct
C6	Solving mathematical problems often entails the use of hypotheses, approaches, tests, and re-evaluations
D1	The pupils learn from seeing different ways to solve a problem, either by pupils presenting their solutions or by the teacher presenting alternative solutions
Please indicate how important each element below is for the pupils (not very important, somewhat important, important, very important)	
C12	The ability to explain their answers
C13	The ability to argue for their procedures and answers
C14	Being able to explain their reasoning
C15	Being able to evaluate other procedures than their own
C16	Being able to follow the reasoning of another pupil
D11	The ability to solve complex problems where the pupils have to use several aspects of mathematics
D12	Teaching must focus on understanding as much as possible so that the pupils can explain methods and connections

The object of this study is mathematical knowledge, so it is the global beliefs that are measured. The focus is on the emphasis that teachers put on different aspects of mathematical knowledge, through statements about what is important for the students to know and what is important to teach. One aspect in the statements is the difference between static and dynamic views of mathematics. This is parallel to the distinction between a characterization of mathematics as content and as a process. An example is found in *Principles and Standards for School Mathematics* (NCTM, 2000), where both content standards (number and operations, algebra, geometry, measurement, data analysis and probability) and process standards (problem solving, reasoning and proof, connections, communication, representation) are defined. The content characterization reflects a more static view while the process characterization reflects a more dynamic view of mathematics. Another aspect in the statements is the difference between different types of understanding, such as instrumental and relational understanding (Skemp, 1976). Instrumental understanding represents the mastering of rules and methods without insight into the reasons that make the rules and methods work. Relational understanding represents an insight into the logic of how methods work and how different parts of mathematics are connected.

Knowledge and Beliefs in this Study

The beliefs constructs are based on a questionnaire consisting of statements, while the knowledge constructs are based on a test. In this way, it is quite clear how knowledge and beliefs are separated in this study. On the other hand, the teachers could regard some of the statements from the questionnaire as knowledge and not beliefs. One example is statement A5 from the rules construct 'if you cram and practice enough, you will get good at mathematics'. Some teachers might regard their answer as based on knowledge because they base the answer on their own experience. For these teachers, it is not a matter of different views or beliefs; it is a matter of knowing what is important for the students. This illustrates how different the distinction is between knowledge and beliefs.

Findings and analysis

Out of the 356 teachers, 54 did not answer one or more questions from the rules construct and 22 did not answer one or more questions from the reasoning construct. To avoid losing too much information, those missing one or two questions in a construct had these missing values replaced with their personal mean for the construct. The rules construct consisted of ten questions and the reasoning construct consisted of eleven questions. Replacing one or two missing questions in constructs of ten and eleven questions is quite moderate. In a study of how to deal with missing data on Likert scales, Downey and King (1998) found that the 'personal mean substitution approach' used here worked well as long as the number of missing respondents did not exceed 15% and the number of missing items did not exceed 20%. The replacements done in this study are within these limits. After this replacement process only six were missing from the rules construct and four from the reasoning construct. Some of these where missing from both constructs, hence the total number of participants with missing items was reduced to seven respondents.

Exploring the Constructs

When considering the correlation between the constructs as presented in Table 3, it is interesting to observe that the rules construct has a negative correlation with all the other constructs. This means that teachers who emphasise rules have a tendency towards a lower score on SCK and CCK, and less emphasis on reasoning, argumentation and justification, and vice versa. Among the rest of the constructs there were no correlations.

Table 3
Correlations between the four constructs

		Reasoning	SCK	CCK
Rules	Pearson Correlation	-.196**	-.245**	-.133*
	Sig. (2-tailed)	.000	.000	.013
Reasoning	Pearson Correlation		.078	.004
	Sig. (2-tailed)		.147	.947

** Correlation is significant at the 0.01 level (2-tailed)
* Correlation is significant at the 0.05 level (2-tailed)
Listwise N=349

Mathematical teaching experience

Mathematical teaching experience generally has an effect on the learning of SCK and CCK. As Table 4 shows the effect is stronger for SCK than for CCK.

Table 4
Mathematics teaching experience and knowledge

		SCK	CCK
Mathematical teaching experience	Pearson Correlation	.267**	.168**
	Sig. (2-tailed)	.000	.001
	N	356	356
Age	Pearson Correlation	-.017	-.130*
	Sig. (2-tailed)	.762	.021
	N	318	318

** Correlation is significant at the 0.01 level (2-tailed)
* Correlation is significant at the 0.05 level (2-tailed)

This effect indicates that the teachers learn some mathematics during their practice as mathematics teachers, and more SCK than CCK. Age could be the real factor here, because teacher education and society have changed. For example, the older teachers were trained when teacher education recruited from the strongest students. But as we can see, age does not correlate with SCK. For CCK there is a small negative correlation with age, which in fact strengthens the interpretation of the positive correlation between mathematics teaching experience and CCK.

Table 5 shows that there are no correlations between mathematical teaching experience and the beliefs constructs. This was important to test because it is possible to imagine that either the reasoning or the rules aspect becomes more or less important for the teachers as they gain experience from real-life teaching. Also Nisbet and Warren (2000) have found that ”beliefs about teaching mathematics are not significantly influenced by the number of years a teacher has been teaching" (p. 41). They suggest that this can be a sign of a lack of systematic professional development for teachers.

Table 5
Mathematics teaching experience and beliefs

		Rules	Reasoning
Mathematical teaching experience	Pearson Correlation	.050	.008
	Sig. (2-tailed)	.353	.887
	N	350	352

To investigate whether or not the correlation between mathematical teaching experience and the knowledge constructs (SCK and CCK) is different with different beliefs, the teachers were divided into three groups in two different ways. One division was based on the reasoning construct (Table 6), and one was based on the rules construct (Table 7). SPSS was used to find cut points as near 33% and 66% as possible. It should not be assumed that the teachers in the lower third of the reasoning group are the same teachers as in the top third of the rules group. Only the lowest and the highest third are presented, as the middle third gave no other information than the fact that its values lie between the two other groups. In Table 6, controlling for age showed no significant correlations.

Table 6
Mathematics teaching experience and different emphasis on reasoning

Reasoning Construct		Lowest third of the teachers		Highest third of the teachers	
		SCK	CCK	SCK	CCK
Mathematical teaching experience	Pearson Correlation	.241**	.207*	.326**	.189*
	Sig. (2-tailed)	.008	.024	.000	.024
	N	119	119	143	143

Comparing Tables 4 and 6 it is clear that for those emphasizing reasoning there is a stronger correlation between mathematical teaching experience and SCK. One possible explanation is that those teachers who emphasise reasoning learn more SCK from their practice than the others. This explanation is plausible because emphasizing reasoning in practice has the consequence that the teacher discusses and argues with the students more often than those who do not emphasise reasoning. As a result, the teacher often needs to assess students' suggestions and arguments, and this type of assessments requires a knowledge that is at the core of SCK. It is possible to explain the higher correlation by the learning of SCK that these teachers obtain from their regular need to assess student thinking in discussions. The strength of the correlation is medium, and the difference from the rest of the teachers is noteworthy but not strong.

There are of course other possible explanations for the correlation. It is possible that those who learn SCK also change their beliefs towards a higher emphasis on reasoning. Some teachers may be more aware of student thinking than others, and through the analysis of student thinking it is possible to learn SCK. It is also possible to learn SCK through a study of elementary mathematics, by focusing on connections and logical explanations. A result may be a stronger focus on students' arguments, and more use of discussions. This could then have an impact on the beliefs about reasoning.

A correlation can never explain the direction of an impact. In this case there are good arguments for both directions as an emphasis on reasoning can result in the learning of SCK, and the learning of SCK can result in a higher emphasis on reasoning. But it may be more reasonable to look at the impact as one that goes both ways. When a teacher learns SCK, it might change his beliefs, and when a teacher changes his beliefs about reasoning, it might result in teaching that promotes the learning of SCK. Thus, SCK and an emphasis on reasoning might strengthen each other. In Table 7 the teachers are divided into three groups based on the rules construct and controlling for age showed no significant correlations. Comparing Tables 4 and 7, it seems like different emphases on rules do not reveal any noteworthy difference for the correlation between mathematical teaching experience and mathematical knowledge.

Table 7
Mathematics teaching experience and different emphasis on rules

Rules Construct		Lowest third of the teachers		Highest third of the teachers	
		SCK	CCK	SCK	CCK
Mathematical teaching experience	Pearson Correlation	.257**	.206*	.260**	.163
	Sig. (2-tailed)	.004	.022	.008	.098
	N	124	124	104	104

Mathematical Education

One hundred and two of the teachers did not report their education in mathematics and in mathematics education. The group of 254 teachers who reported their education was compared to all 356 teachers in order to investigate whether the 254 teachers represent the same population as the whole group does. Background variables such as age, gender, experience (general teaching experience, mathematics teaching experience, and experience in teaching at different grades), actual teaching grade, education (mathematics and mathematics education), and the constructs of rules, reasoning, CCK and SCK were compared. There were differences, but these were small. The conclusion is that the group of 254 teachers who reported their education represents the same population as the total of 356 teachers who participated. Table 8 presents the correlation between mathematics education (measured by ECTS in mathematics and in mathematics education) and the four constructs. ECTS (European Credit Transfer and Accumulation System) is a European standardisation based on the convention that 60 credits represents the workload of a full-time student during one academic year. A full-time study program in Europe is in most cases from 36 to 40 weeks per year, which means that one credit stands for 24 to 28 working hours. Most mathematics courses registered in this research were equivalent to 30 credits, while some older courses were equivalent to 15 credits.

Table 8
Mathematics education and the four constructs

		SCK	CCK	Rules	Reasoning
Mathematics education	Pearson Correlation	.320**	.266**	-.174**	.072
	Sig. (2-tailed)	.000	.000	.006	.256
(ECTS in mathematics and in mathematics education)	N	254	254	251	252

There is a general correlation between the knowledge constructs (SCK and CCK) and the teachers' education. This indicates that the more mathematics education a teacher has, the more the teacher knows of SCK and CCK. It is also possible that those teachers who already possess a high degree of mathematical knowledge are exactly those teachers who want more education. This is of course not surprising. Also, there is a small negative correlation between education and the rules construct, meaning that there is a tendency for teachers to emphasise rules less the more mathematical education they have.

In order to investigate whether or not the correlation between mathematical education and the knowledge constructs (SCK and CCK) is different with different beliefs, the three groups based on the reasoning construct (Table 9) and the three groups based on the rules construct (Table 10) were used. There are some differences in education between these groups. On average, the third of teachers whose emphasis on reasoning is low have 28 ECTS while those emphasising reasoning have 33 ECTS. The difference is opposite when the data are divided based on the rules construct. Here the third with a low emphasis on rules have 36 ECTS on average, while the third with a high emphasis on rules have 27 ECTS.

Comparing Tables 8 and 9 it is clear that different emphases on reasoning result in different correlations between mathematics education and mathematical knowledge. Within the group that emphasises reasoning, there is a slightly stronger correlation between education and SCK and a weaker (and not significant) correlation between education and CCK. The low and not significant correlation between education and CCK could be explained if this group generally had a high level of CCK. But the mean and standard deviations show no significant difference from the other groups.

Table 9
Mathematics education and different emphasis on reasoning

Reasoning Construct		Lowest third		Highest third	
		SCK	CCK	SCK	CCK
Mathematical education	Pearson Correlation	.237*	.357**	.357**	.184
	Sig. (2-tailed)	.024	.001	.000	.066
(ECTS in mathematics and in mathematics education)	N	91	91	100	100

The most interesting observation is that there is a difference between the correlations of education with SCK and CCK (Table 9). One explanation might be that emphasizing reasoning has an impact on the learning outcome from the education. This interpretation is based on the view that beliefs act as filters through which one sees the world. Teachers who emphasise reasoning probably more often connect the mathematics they learn to the situations they expect to (or know they will) meet in practice. It may also be that aspects dealing with reasoning, argumentation and justification are considered more important by these teachers and as a result are learned better. On the other hand, rules and formal mathematics might not be considered equally important. There are good reasons to believe that an emphasis on different aspects of a course and different ways to relate the course to practice will result in different knowledge.

Another possible explanation may be that some teachers learn more SCK and less CCK during their education, for some unknown reason. Because SCK and an emphasis on reasoning are connected theoretically, these teachers may end up emphasizing reasoning as a result of their knowledge. Also here, it is possible that the learning of SCK and an emphasis on reasoning are strengthening each other.

Within the group that does not emphasise reasoning, the results are almost the inverse. Comparing Tables 8 and 9, it is clear that there is a stronger correlation between education and CCK and a weaker correlation between education and SCK. The same interpretations as above are also possible here;

either the teachers learn different things because of their beliefs, or they do not emphasise reasoning as a result of their knowledge. It is also possible that CCK and a lack of emphasis on rules are strengthening each other.

Comparing Tables 8 and 10, it is clear that different emphases on rules result in different correlations between mathematics education and mathematical knowledge. Within the group that emphasises rules, there is a stronger correlation between education and CCK and a weaker correlation between education and SCK (than for all the teachers in table 6). Within the group that does not emphasise rules, the situation is reversed, as there is a stronger correlation between education and SCK and a weaker correlation between education and CCK.

Table 10
Mathematics education and different emphasis on rules

Rules Construct		Lowest third		Highest third	
		SCK	CCK	SCK	CCK
Mathematical education	Pearson Correlation	.378**	.161	.181	.333**
	Sig. (2-tailed)	.000	.138	.118	.003
(ECTS in mathematics and in mathematics education)	N	86	86	76	76

As above, this can be interpreted in three different ways. It may be that the beliefs have an impact on what you learn from your education, or it may be that your knowledge forms your beliefs. Most likely, however, knowledge and beliefs influence each other.

Conclusion

In the group that emphasises reasoning there is a stronger correlation (compared to the whole group) between SCK and both mathematical teaching experience and mathematics education. This could be because those emphasizing reasoning learn more SCK, or it could be because learning SCK has an impact on the emphasis on reasoning. Probably the answer is both; that SCK and an emphasis on reasoning strengthens each other. This means that an emphasis on reasoning is an affordance for the learning of SCK, and that the learning of SCK is an affordance for an emphasis on reasoning (or a barrier against not emphasizing reasoning).

For two groups, the results are similar; the group that does not emphasise reasoning and the group that emphasises rules. In both groups there is a stronger correlation (compared to the whole group) between CCK and mathematics education, and a weaker correlation between SCK and mathematics education. Also here, it could be the beliefs that cause the difference in knowledge, or it could be the difference in knowledge that causes the beliefs. If the impact goes both ways, then both a lack of emphasis on reasoning and an emphasis on rules seems to act as an affordance for the learning of CCK and a barrier against the learning of SCK from education, while learning more CCK and less SCK (than the whole group) from education seems to act as an affordance for not emphasizing reasoning and for an emphasis on rules (or as a barrier against an emphasis on reasoning or against not emphasizing rules).

The group that does not emphasise rules has some similarities with the group that emphasises reasoning. Both groups show the same results for education, but the group that does not emphasise rules does not show any noteworthy differences from the whole group regarding mathematical teaching experience. If the impact goes both ways, then not emphasizing rules acts as an affordance for the learning of SCK and as a barrier against the learning of CCK, while learning more SCK and less CCK (than the whole group) acts as an affordance for not emphasizing rules (or as a barrier against an emphasis on rules).

This study has provided information about the interplay between beliefs and knowledge. In a similar study by White, Way, Perry and Southwell (2006), little interplay is found, and they stated in the conclusion that "a more comprehensive instrument in mathematical achievement is needed" (p. 46). Their mathematical instrument was an achievement test of the mathematics the teachers would be expected to teach, from early primary to lower secondary school. This is quite similar to the CCK construct used in this study.

In this study, SCK is added as a second knowledge construct. One important condition for the findings in this study is the division between two different knowledge constructs. The most important findings are that different emphases on beliefs are connected to different aspects of mathematical knowledge. By choosing only CCK or SCK, the results would only expose positive and negative differences related to beliefs. A consequence of these results is that there is a need to consider beliefs and knowledge not only as connected, but as elements that strengthen each other. This would also be important to address in teacher education.

We do know that beliefs and knowledge influence practice. Wilson and Cooney (2002) state that

> … regardless of whether one calls teacher thinking beliefs, knowledge, conceptions, cognitions, views, or orientations … the evidence is clear that teacher thinking influences what happens in the classrooms, what teachers communicate to their students, and what students ultimately learn (p. 144).

But the limitation on research not involving classroom data is that it is not possible to know the extent to which the espoused beliefs (from the questionnaire) are also enacted in the teachers' classrooms, or how knowledge has an impact on practice. For example, the reasoning construct represents the emphasis that the teacher puts on reasoning, argumentation and justification. It is difficult to know whether or to what extent this emphasis will be evident in the teacher's practice. If the reasoning construct represents beliefs that are psychologically central to the teachers there are reasons to believe it will often influence the teaching, while this will happen less often if the beliefs are more peripherally held. For this reason, the connection between the teachers' espoused beliefs (the questionnaire) and their enacted beliefs (in practice) are different from teacher to teacher. This means that some teachers who emphasise reasoning in the questionnaire are not doing so in practice, and are probably not learning as much SCK from discussions as those who also emphasise reasoning in practice.

In order to really find out whether the teachers who emphasise reasoning show a stronger correlation between mathematical teaching experience and SCK, it is necessary to observe the teachers and select those who do what they say. Consequently, there is a need for a better understanding of how these constructs play out in the classroom. This may also help us understand more about why certain aspects of knowledge are connected to certain types of beliefs.

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