Student Teachers' Conceptions of Fractions: A Framework for the Analysis of Different Aspects of Fractions

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Fractions are core content of elementary school mathematics, and conceptual knowledge of fractions is essential when developing a comprehensive understanding of fractions. Previous research, however, has indicated limitations in student teachers' fraction knowledge. This study investigated 57 Swedish elementary school student teachers' conceptions of fractions. The data were collected using a paper-and-pencil questionnaire and analysed with an analytical framework building on previous research on four core components of fractions. Using the devised analytical framework, we were able to characterise the conceptual content shown in the student teachers' answers and identify gaps in their fraction knowledge. The most severe gaps were identified in relation to interpretations of fractions, where only the part-whole and the quotient interpretations were identified; the measure, operator, rate, ratio, and number interpretations were missing completely. Aspects of fractions related to representations and procedures were better represented in the participants' conceptions of fractions, but we also illustrate substantial differences between the student teachers. In addition to this quantitative description, we provide qualitative examples. The results raise some questions and implications to be addressed in teacher education programs when developing student teachers' fraction knowledge.

Keywords · conceptions of fractions · conceptual knowledge · elementary school · student teachers · teacher education

Background and Aim

It is expected that student teachers acquire the mathematical knowledge necessary to effectively teach elementary school mathematics, including the core content of fractions, during their teacher education. Research, however, has suggested that fractions is among the most challenging areas in school mathematics to learn and teach (Ibañez & Pentang, 2021; Lamon, 2007). Effective instruction of fractions requires teachers to be aware of common student errors and to understand how the errors relate to fundamental mathematical concepts (Bray, 2011). Despite the importance of this knowledge, numerous studies have highlighted limitations in student teachers' fraction knowledge (e.g., Marchionda, 2006; Newton, 2008; Young-Loveridge et al., 2012). While student teachers may demonstrate proficiency in performing fraction procedures, their knowledge of fractions' procedural and conceptual aspects is often deficient (Lin et al., 2013; Lovin et al., 2018; Muir & Livy, 2012; Young & Zientek, 2011). Procedural knowledge of fractions concerns computational skills related to fraction tasks and familiarity with the proper ways to denote fractions and their operations, whereas conceptual fraction knowledge includes understanding the connections between fractions and other mathematical constructions (Hiebert & Lefevre, 1986).

Student teachers often enter their teacher education with fraction knowledge that mainly concerns procedures and is limited in its conceptual basis (Chinnappan & Forrester, 2014). Moreover, their fraction knowledge does not seem to improve much during teacher education (Lin et al., 2013; Lovin et al., 2018; Van Steenbrugge et al., 2014). Stohlmann et al. (2014) found that at the beginning of a mathematics and pedagogy content course, student teachers focus on procedural fluency, and they believe that conceptual understanding of mathematics is not more powerful or generative than remembering mathematical procedures. Also, final-year student teachers have difficulties creating connections among mathematical concepts that might be needed for the creation of meaningful conceptual representations when teaching fractions (e.g., Tirosh, 2000). Previous research has shown that those already working as mathematics teachers also have difficulties with fractions, for example, when conceptualising the division of fractions (Lamberg & Wiest, 2015). Juter (2022) found that student teachers have difficulties with their arguments for mathematical structures and representations of concepts in real number contexts, stating that this kind of incoherent mathematical content knowledge and concept image can hinder teaching mathematics meaningfully. Thus, procedural-based fraction knowledge may have limited value in mathematics teaching, and it may even impede the development of what Ball et al. (2008) call specialised content knowledge and pedagogical content knowledge needed to teach fractions effectively (Chinnappan & Forrester, 2014).

Differences in mathematics teachers' teaching performance are often due to differences in their knowledge of concepts and connections, which may lead to rule-based procedural teaching (Tchoshanov, 2011). Many student teachers' fraction knowledge is rule-based and includes incorrect memories of algorithms they have learned before attending teacher education (Bansilal & Ubah, 2020; Jóhannsdóttir & Gísladóttir, 2014). When enhancing elementary school students' mathematics learning, however, teachers are supposed to use effective teaching practices that also improve their students' conceptual thinking. This kind of effective teaching consists of practices that make connections and generate conceptual discourses (Anghileri, 2006). Developing effective teaching practices in mathematics indicates to student teachers the importance of gaining knowledge of fraction concepts and fraction connections during teacher education. Moreover, student teachers' conceptual knowledge needs to be investigated further to ensure their limitations are addressed so that they do not pass on their conceptual misunderstandings to elementary students later in their professions (Juter, 2022).

The study reported in this article responds to the research needs in the field of mathematics education focusing on elementary school student teachers' conceptions of fractions. In the context of this study, elementary school consists of Grades 1–6, that is children 6–12 years of age. While Charalambous and Pitta-Pantazi (2007) used Behr et al.'s (1983) theoretical model to investigate elementary school students' understanding of fraction interpretations, the current study contributes to the field by addressing the topic in a Swedish teacher education context with a wider range of fraction-related aspects than what was considered by Behr et al. (1983). The study aimed to investigate how student teachers demonstrate their conceptual knowledge of fractions in a paper-and-pencil questionnaire when asked to relate the concept of fractions to other concepts and mathematical constructions. To support the research, a new framework for analysing different aspects of fractions was developed. The framework is used in this article to describe the fraction aspects that the participating student teachers refer to and analyse gaps in student teachers' conceptual fraction knowledge. The devised framework and its background will be presented in the Theoretical Perspectives and Methodology sections of this article. The study was designed to answer the following research questions:

RQ1: Which aspects of fractions do student teachers refer to in their conceptions of fractions?

RQ2: What gaps related to the aspects of fractions can be identified in student teachers' conceptions of fractions?

Theoretical Perspectives

This section is organised in two parts. First, we give a basic overview of ideas related to conceptual knowledge and its application to fractions. Then, we present an overview of the research literature on fractions, which we organise into four themes that will later form the basis of our analytical framework.

Conceptual Knowledge and Fractions

Mathematics understanding is often categorised as two distinct approaches, conceptual and procedural knowledge. The former is the knowledge that is rich in relationships (Hiebert & Lefevre, 1986), for example, understanding the definition of fractions and how the fundamental facts of different number sets are related to fractions. While it is not always easy to draw a strict line between conceptual and procedural knowledge, in the context of fractions, procedural knowledge can relate to computational skills and using established rules and notations for fraction operations. When describing the relationship between conceptual and procedural knowledge, however, Hiebert and Lefevre (1986) stated, "… the problem is that the types of knowledge themselves are difficult to define. The core of each is easy to describe, but the outside edges are hard to pin down" (p. 3).

Conceptual and procedural knowledge can also be seen as closely interrelated and intertwined and build on each other (Thurtell et al., 2019). Moreover, many researchers have suggested that concept formation develops through a procedural approach to conceptual understanding (e.g., Gray & Tall, 1994). Sfard (1991) concluded that the transition from computational operations to abstract objects is a difficult process consisting of three hierarchical stages: interiorisation (a learner is acquainted with performing operations of lower-level mathematical objects), condensation (the learner becomes capable in alternating between different representations of a mathematical concept; the new concept is born), and reification (the learner conceives the mathematical concept as a fully-fledged object; the concept is reified). At the first stages of concept formation, Sfard (1991) asserted an operational conception is developed. This type of conception of an abstract mathematical notion is conceived as a product of a certain process, and it is supported by verbal representations and seen as necessary, but may not be sufficient for effective learning. From the operational conception evolves a structural conception, which characterises the mathematical notion as a static structure, an object. Visual images often support this conception and facilitates learning. In the case of fractions, the operational description is about seeing fractions as a result of a division of integers whereas the structural description considers fractions as a pair of integers (Sfard, 1991). Not all learners, however, reach the reification stage—which brings relational understanding—because reification demands much effort and the ability to see something familiar in a new way (Sfard, 1991). As Sfard (1991) stated, "… pupils can be quite successful in computations involving fractions in spite of being unable to treat fractions as numbers" (p. 32). This may also concern student teachers' conceptual knowledge of fractions. For example, Siegler and Braithwaite (2017) noted, "Although written fraction notation is usually introduced in early elementary school, connecting written fractions with the magnitudes that they represent remains challenging even for many adults" (p. 195).

Hallett et al. (2010) suggested that children's use of conceptual and procedural knowledge is based on their individual differences more than on their developmental processes, and that they seem to use a combination of conceptual and procedural knowledge when working with fraction tasks. When learning fractions, however, children who rely on their conceptual knowledge seem to have an advantage compared to those who rely solely on procedural knowledge. Pantziara and Philippou (2012) conducted a study investigating sixth grade students' conceptualisations of fractions using the three stages proposed by Sfard (1991). Pantziara and Philippou focused especially on the part-whole interpretation and the measure interpretation of fractions. They also found that students who rely only on procedural knowledge had lower performance on fraction tasks than those who apply conceptual knowledge as well. Previous research has revealed several deficiencies in student teachers' conceptual knowledge of fractions (e.g., Ibañez & Pentang, 2021). Student teachers' limited conceptual knowledge of fractions can be seen, for example, in their difficulties solving and creating fraction word problems and understanding the meanings behind fraction procedures (López-Martín et al., 2022; Marchionda, 2006; Olanoff et al., 2014; Toluk-Uçar, 2009).

Conceptual understanding in the learning of mathematics is demonstrated by making translations from one mathematical representation to another, which is challenging because representations are something that stands for something else (Duval, 2006). Previous research has shown that student teachers lack flexibility when using conceptual fraction knowledge with representations other than mathematical algorithms to demonstrate fraction operations (e.g., Lee & Lee, 2023; Olanoff et al., 2014). The role of different representations, however, is essential in the teaching of mathematics. For example, when learning complex content such as fractions, there seem to be benefits in using multiple representations and making links among them (Ainsworth, 2006; Graeber, 1999; Thurtell et al., 2019). Cramer et al. (2002) stated that using multiple representations is effective in the learning of rational number concepts and procedural skills with fractions. Moreover, as Ebbelind et al. (2012) showed, the interplay between spoken language, gestures, iconic images (pictures), and concrete materials help elementary school students to solve symbolic fraction tasks. Similarly, a representation-based teaching model seems to develop student teachers' conceptual knowledge of fractions (Thurtell et al., 2019). It has also been shown that teachers' knowledge of drawn representation models of fraction multiplication and division is strongly connected to their motivation and instructional practices for using such models (Jacobson & Izsák, 2015).

Conceptual understanding of fractions can also be regarded as mastering different interpretations of fractions (Behr et al., 1983; Cramer et al., 2002). As Vergnaud (2009) stated, the meaning of a concept originates from various situations based on systems of several concepts. Building on the work of Vergnaud (2009), Ahl and Helenius (2021a, 2021b) discussed the complication that arises when the same concept has several but related meanings and call it conceptual polysemy. Ohlsson (1988) wrote about the difficulty of fractions as follows:

> How should fractions be understood? The complicated semantics of fractions is, in part, a consequence of the *composite nature* of fractions. How is the meaning of 2 combined with the meaning of 3 to generate a meaning for 2/3? The difficulty of fractions is also, in part, a consequence of the bewildering array of *many related but only partially overlapping ideas* that surround fractions. What are the relations between fractions, measures, proportions, quotients, rates, ratios, rational numbers, and so on? (p. 53)

Further, Tall and Vinner (1981) used the term "concept image to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (p. 152). In the case of fractions, the difficulty is that the concept image of many learners is not coherent, and thus, they focus on memorising separate rules for fraction procedures rather than understanding the fraction concept and its relations to other concepts (e.g., Joutsenlahti & Perkkilä, 2021; Lortie-Forgues et al., 2015). This study focuses on the participants' conceptions that represent their individual cognitive structure of the fraction concept but not the formal definition of the concept, as discussed by Tall and Vinner (1981).

Different Aspects of Fractions

In this section, different aspects of fractions are decribed according to four themes. We start with the selection of interpretations of fractions. Then, we go over the theme of representations of fractions, which involves characterisations of verbal, visual, and symbolic representations. The third aspect is procedures related to fractions involving, among other things, operations such as addition and multiplication, as well as procedures for reducing and extending fractions. Finally, we present an aspect that pragmatically combines additional notions and concepts related to fractions.

Interpretations of fractions

Fractions have been interpreted differently by researchers based on the connections between fractions and other mathematical constructs and the situations where fractions are used. The complexity of learning and teaching fractions is highly connected to the multifaceted construct of fractions (Kieren, 1993; Lamon 2020). Some researchers also call the fraction interpretations subconstructs, and the content for interpretations and subconstructs is defined slightly differently depending on the researcher (e.g., Agathangelou & Charalambous, 2021; Charalambous & Pitta-Pantazi, 2007). This study uses the term interpretation for the interpretations and subconstructs found in previous research.

Charalambous and Pitta-Pantazi (2007) refered to Kieren (1976) as the first to conceptualise the concept of fractions as a set of interrelated constructs that includes the interpretations of part-whole, ratio, operator, quotient, and measure. Understanding these five core interpretations of fractions can be regarded as a prerequisite for solving problems in the fractional number domain (Charalambous & Pitta-Pantazi, 2007). The part-whole interpretation, however, is typically seen as the most essential and a fundamental construct when developing the rational number concept and understanding of the multiple meanings of fraction interpretations, and is related specifically to the other interpretations (Behr et al., 1983; Charalambous & Pitta-Pantazi, 2007).

Within school mathematics, fractions are often mixed up with positive rational numbers, even though a full appreciation of the formal construction of the rational number system, as it is done in formal mathematics, is beyond the grasp of school students (Thompson & Saldanha, 2003). Ball (1993) summarised fraction interpretations from several studies and stated that as rational numbers, fractions can be

> interpreted (a) in part-whole terms, where the whole unit may vary, (b) as a number on the number line, (c) as an operator (or scalar) that can shrink or stretch another quantity, (d) as a quotient of two integers, (e) as a rate, or (f) as a ratio. (p. 168)

Some researchers also use the term fraction model in connection to fraction interpretations (e.g., Behr et al., 1983; Lamon, 2020; Lee & Lee, 2023). In area models, one whole (the unit) is a given area, which is partitioned to form fractions. Fraction strips, circles, food such as cakes and pizzas, and Cuisenaire rods are examples of area models whereas candies, coins, and two-colour counting chips are examples of discrete (set) models, where individual objects or sets of objects form the unit whole (Lamon, 2020). Thus, unlike whole numbers, where one means one object, the reference unit in fractions may include more than one object or several objects packaged as one. The part-whole interpretation is considered as a situation where a continuous quantity or a set of discrete objects is partitioned, that is, divided up into equal size parts (Behr et al., 1983; Lamon 2020; Marshall 1993), and a fraction is a relative amount telling how much you have relative to the unit. For example, the fraction $\frac{2}{4}$ can be conceived as a part of a whole, that is, two out of four equal parts. When using the part-whole interpretation, it is important to understand the relationship between the numerator and denominator rather than using them as whole numbers.

The part-whole interpretation is most usually emphasised in school mathematics; it is the interpretation "that students encounter the longest and meet more frequently in their mathematics textbooks" (Charalambous & Pitta-Pantazi, 2007, p. 309). In the context of this study, the part-whole interpretation is also emphasised by the national curriculum, and thus, most probably formed already during the very first school years (Skolverket, 2022a). Moreover, students in elementary grades are supposed to learn how the parts are named and expressed as simple fractions, such as $\frac{1}{2}, \frac{1}{4}, \frac{1}{5}$, and $\frac{2}{3}$ (Skolverket, 2022a, 2022b). The part-whole relationship, however, can be a hindrance when dealing with fractions larger than the whole number 1 (Rønning, 2013). Lovin et al. (2018) showed that the focus on using the part-whole model can also be an obstacle for more advanced fraction schemes.

When using a length model, also called a linear model, fractions can be interpreted as numbers on the number line. Lamon (2020) also refered to the linear model as a measure interpretation that conveys how big the fractions are and sets them to a point on a number line. A fraction such as $\frac{3}{4}$ can be seen as a unit fraction that is used repeatedly to determine a distance from a given starting point, $\frac{3}{4}$ then corresponds to the distance of 3 ($\frac{1}{4}$ units) from that point (Charalambous & Pitta-Pantazi, 2007; Marshall, 1993). The linear number line model of fractions, however, is a difficult model for many elementary students because the measure interpretation is not emphasised as frequently as the part-whole contruct in mathematics lessons (Charalambous & Pitta-Pantazi, 2007).

Fractions as operators can be shrinking and stretching other quantities, like $\frac{1}{2}$ x 8 = 4 and $\frac{5}{2}$ x 6 = 15. This interpretation is useful, for example, when investigating fraction equivalence and working with the multiplication operation (Behr et al., 1983). Further, the quotient interpretation is fraction as division. A fraction such as $\frac{2}{4}$ can be conceived as a quotient representing the result of division (two divided by four). The ratio interpretation differs from the part-whole and quotient interpretations since it does not involve the idea of partitioning. Rather, the ratio construct refers to the comparison between two quantities (e.g., two parts to four parts), and the rate interpretation can be used as an extended ratio in situations where two quantities define a new quantity, for example, speed is a relationship between distance and time (Behr et al., 1983; Lamon, 2020).

Understanding the different fraction interpretations decribed above is essential for deep fractional understanding. Getenet and Callingham (2021) concluded that how students think about fractions is deeply influenced by how the fraction interpretations are taught by teachers. Previous research, however, has shown that student teachers have difficulties understanding the various fraction interpretations and that they mostly prefer the part-whole interpretation (Olanoff et al., 2014).

Representations of fractions

In addition to the different fraction interpretations, fractions can be represented in several ways. In school mathematics, using different verbal, visual, and symbolic representations for fractions is common. Verbal representations are expressed in spoken and written language, for example, "two thirds" and "one and a half". Verbal expressions are also dependent on the interpretation used for a fraction. For example, according to the part-whole interpretation, the fraction $\frac{1}{4}$ can be expressed as "one fourth" but when interpreted as a ratio, the same fraction can be expressed several ways like "one for four" and "the ratio of one to four". For visual representations, both drawings and pictures (e.g., pie charts, number lines) and manipulative aids (e.g., Cuisenaire rods, pattern blocks, fraction bars and fraction circles, multilink cubes, folded paper) can be used to illustrate and concretise the notion of fractions.

The different representations used to express fractions symbolically make it more difficult to understand fractions than whole numbers (Lortie-Forgues et al., 2015). In symbolic representations, fractions are written using two numerals, but a fraction stands for one number that includes three parts: a numerator (the top number in the fraction form $\frac{\mathrm{a}}{\mathrm{b}}$), a denominator (the bottom number), and a fraction line (the viniculum) that separates these two numbers, for example, $\frac{1}{4}$. Different fractions can represent the same fractional quantity, for example, $\frac{1}{2}$ is equal to $\frac{2}{4}$ and $\frac{5}{10}$. Fractions can also be addressed with the form a/b, that is, 1/4 = $\frac{1}{4}$, or used in the form of a mixed number as well. An improper fraction converted to a mixed number includes a whole number part and a fraction part, for example, $\frac{4}{3} = 1\frac{1}{3}$.

Even though decimals and percentages are often seen as separate topics in elementary school mathematics, they are special kinds of fractions with their own forms of notation. For example, $\frac{1}{4}$ = 0.25 = 25%. Formally, 0.25 stands for 2 tenths ($\frac{2}{10}$) and 5 hundredths ($\frac{5}{100}$) or 25 hundreths ($\frac{25}{100}$) in total, which makes 25% an alternative representation of $\frac{25}{100}$. Moreover, mathematics learning can be enhanced if learners are able to make connections among fractions, decimals and percentages, understanding that numbers in these forms may represent the same part of a whole and that some calculations can be proceeded with different solution methods when using fractions, decimals, or percentages (Anghileri, 2006). Previous research has shown that student teachers have difficulties converting fractions to decimals and seeing decimals as also having a meaning that is connected to size and quantity (Muir & Livy, 2012).

Using different representation forms, concrete materials, diagrams, symbols, and the knowledge of the connections between fractions, decimals, and percentage forms can be regarded as core requirements in the learning of mathematics (Skolverket, 2022a). Based on their findings with sixth grade students, Pantziara and Philippou (2011) suggested that the use of different representations and the alternation between the representations are substantial factors for the development of students' conceptual knowledge of fractions. Lee and Lee (2023), however, found that some student teachers' use of invalid representations with fraction addition indicated a misunderstanding of the key concepts of fractions. Moreover, previous research has also revealed student teachers' difficulty in making sense of fraction solutions different from their own, which indicates limitations in their use of different representations (e.g., Jakobsen et al., 2014).

Procedures related to fractions

Fractions can also be used with several mathematical procedures. Mastering the different fraction interpretations and representations presented above usually contributes to performance in fraction operations with addition, subtraction, multiplication, and division, and in fraction equivalence (Charalambos & Pitta-Pantazi, 2007). For example, multiplication and division in fraction tasks can be illustrated using an area model and also solved with decimals (Lamon, 2020). The division and multiplication operations for fractions, however, have shown to be the most challenging for many student teachers (e.g., Newton, 2008; Son & Lee, 2016; Tirosh, 2000).

Procedures like reducing and extending fractions can be considered as prerequisite skills for the addition and subtraction of fractions (Löwing, 2016). Moreover, performing fraction arithmetic requires that a learner can make appropriate fraction connections, determine fraction size, order, and equivalence, as well as master simplifying fractions, converting fractions to mixed numbers and mixed numbers to fractions (Lamon, 2020; Lortie-Forgues et al., 2015). Understanding fraction order and equivalence requires understanding of the relation between the size and number of equal parts in a partitioned unit (Behr et al., 1984). The ability to demonstrate the size of fractions is fundamental in developing rational number concepts, relations, and operations (Behr et al., 1983), and requires knowledge of the multiplicative relationship between numerators and denominators. Further, comparing the size of fractions can be done in different ways, for example, using different ordering strategies: (1) same-size parts, (2) same number of parts, and (3) comparison to a benchmark (Lamon, 2020). Even though the presented procedures can be seen as basic tools when working with fractions, many student teachers possess similar misconceptions of the fraction-related procedures as the children they might be teaching (e.g., Jakobsen et al., 2014; Muir & Livy, 2012; Van Steenbrugge et al., 2014). Moreover, student teachers have difficulties judging their abilities to correctly perform fraction operations (Young & Zientek, 2011).

Further notions related to fractions

The last theme in our framework is formed pragmatically since research literature dealing with fractions also includes notions, terms, and expressions for important fraction-related concepts that do not directly fall into any previous themes. In the case of fractions, this conceptual discourse includes the meaning and use of different and same numerators and denominators, knowledge of equivalent fractions, as well as understanding fraction notations and the use of representations such as fraction line, a unit fraction, a mixed number, and inverted fractions (Löwing, 2016). A common misconception among student teachers, however, is related to the use and meaning of numerators and denominators (Newton, 2008; Young & Zientek, 2011). Moreover, the notion of least common multiples is needed when processing different fraction procedures, especially when extending fractions with addition and subtraction. Fractions should also be understood as numbers in the rational number set.

Tobias (2013) examined student teachers' conceptions and development of language use for describing fractions. Her findings showed that student teachers struggle with understanding the language for defining a whole, and that they have difficulties distinguishing between the questions, *How much ...?* and *How many ...?* in relation to fractional numbers.

The four aspects of fractions presented above can also be considered as a prior knowledge and foundation for algebra and other areas in mathematics. In this study, the different aspects, that is, interpretations, representations, procedures, and notions, form the categories for the analytical framework, which will be described in detail in the next section.

Methodology

Context of the Study

The current study was conducted in connection to one Swedish teacher education institute. In Sweden, elementary teacher education prepares teachers for the preschool class and Grades 1–6 of the nine-year compulsory school. Elementary teachers teach several subjects to children between the ages of 6 and 12, but they are usually, not specialists in their subjects. The core school subjects studied in elementary teacher education are Swedish, English and Mathematics.

The participants of the study were 57 student teachers in the third year of their four-year academic studies. They were at the start of their second mandatory mathematics course that had a focus on mathematics teaching. In the previous year, they had passed successfully a mathematics content course where, for example, fraction content included in the national compulsory school curriculum was addressed. At the time the study, the participants were expected to have recalled and rehearsed fraction concepts, operations, algorithms, and notions studied prior to commencing university studies and also to have deepened their knowledge of fractions from the point of view of teacher education studies.

Data Collection and Analysis

Altogether, 67 student teachers were enrolled in the second teacher education mathematics course. At the time of the data collection, 61 of them were at the university campus where they were asked to answer a paper-and-pencil questionnaire before a mathematics lecture. The voluntary respondents were given 90 minutes to anonymously answer the questionnaire that was comprised of three sections concerning fractions in elementary mathematics teaching. None of the respondents needed the whole reserved time, but four of them chose not to complete the section concerned in this study, thus resulting in 57 participants for the study.

The study used data from the first section where the instructions asked the respondents to reflect on different concepts and connections that they thought might relate to fractions and then write and draw everything they knew about the concept of fractions. In the second section, the participants were asked to describe how they might teach the task $\frac{1}{2} + \frac{3}{4}$ to elementary school students. The last section in the questionnaire involved six routine fraction tasks without context to be solved by presenting all solution steps (for more information, see Tossavainen, 2022). The short background information section was comprised of statements focusing on the participants' experiences on fractions and mathematics teaching. Other questions asked about the participants' expectations concerning their future profession as mathematics teachers and their studies prior to the second teacher education mathematics course. Only the data related to fractions are reported in this paper.

The data were analysed using an analytical framework formed to operationalise the core aspects of fractions. These aspects were found in previous studies and described above in the theoretical section. The aspects are the basis of the four fraction categories in the analytical framework: (F1) Interpretations, (F2) Representations, (F3) Procedures, and (F4) Notions. Each category F1–F4 has several subcategories with different content, described below (see Figure 1).

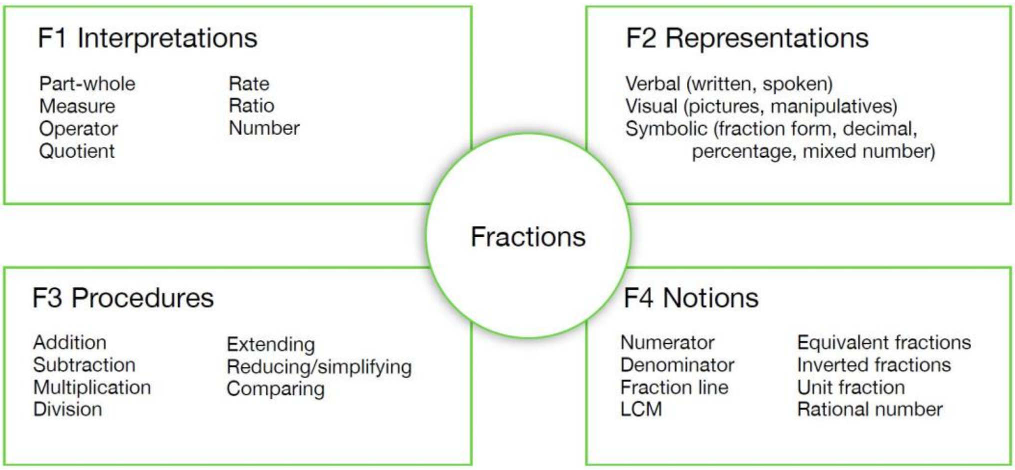


Figure 1. The categories and their subcategories in the framework for analysing different aspects of fractions.

The F1 category includes interpretations of what fractions are. The category F2 focuses on how fractions can be "seen" in different representations when using verbal, visual and symbolic forms of expressions. The F3 category is about how to work with fractions using different mathematical procedures, and the last category, F4, includes core notions that characterise fractions.

To conduct the analysis using the framework, the subcategories and their content in each category were defined more precisely. Thus, F1 includes fraction descriptions that refer to the interpretations presented by Ball (1993) and Lamon (2020): part-whole, measure, operator, quotient, rate, ratio, and number. Descriptions categorised into F2 include written notions for a fraction form, a decimal form, or a percentage, as well as symbolic forms for these representations such as $\frac{3}{4}$, 0.25, and 50%. Moreover, the forms $\frac{a}{b}$ and a/b provided with some procedures or with some written text, for example, $\frac{\text{numerator}}{\text{denominator}}$, were also defined as illustrating the fraction form. The written fraction form in the example, however, includes two core notions of fractions, and thus, such fraction descriptions were defined to belong to the category F4 as well. Both drawings and written descriptions of concrete things and manipulatives, such as circles, cubes, and pizzas were counted as visual representations in F2. The F3 category was defined to include mention of different fraction operations in written texts and descriptions that give examples of these operations with addition, subtraction, multiplication, or division, for example, $\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$. The same concerns the procedures of extending, reducing, simplifying, and comparing fractions. F4 includes written texts expressing the following notions: numerator, denominator, fraction line, least common multiple (LCM), rational number, unit fraction, equivalent fractions, and inverted fractions. In this study, these notions are considered such core concepts that elementary school mathematics teachers are expected to be able to understand and use them in their teaching of fractions. Moreover, the choice of these notions is also justified by their appearance in the Swedish school context. For example, the notion of unit fraction is not used in all languages. Further, the focus of the study is not on the formal definition of the fraction concept; thus, the participants were not asked to give that kind of definition for fractions. Although the categories F1–F4 in the analytical framework can be regarded as relatively distinct, some fraction descriptions may include content for several categories as shown above in the text-based example, $\frac{\text{numerator}}{\text{denominator}}$.

After the analysis, the participants' questionnaire answers were compared to the categories in the analytical framework and a summary of their conceptions of fractions was illustrated with a figure similar to the framework (see Figure 2). When answering the first research question, the analysis focused on the categories and subcategories that the student teachers referred to in their descriptions. For the second research question about the gaps in the student teachers' conceptions of fractions, the analysis focused on the content of the different categories that did not appear in the participants' answers. Moreover, the participants' answers were also analysed qualitatively to investigate their mistakes and misconceptions. The results of the study are reported qualitatively as well as quantitatively in terms of the number of participating student teachers providing a description that belonged to a specific category or subcategory. In the results section, the participants are referred to with the number code in their paper-and-pencil questionnaire, and the excerpts of their written texts have been translated into English.

Results

In this section, we first present some general findings and also give a detailed presentation of the results related to the two research questions in connection to the categories F1–F4. We illustrate the findings with a figure similar to the figure presented in the methods section and we also include some qualitative examples of how the participants referred to the different subcategories. Then, we give an overview of the results summarising them in a more detailed figure at the end of the section.

In general, the four categories for the different aspects of fractions were all represented in the participants' conceptions of fractions (see Figure 2). The difference between the categories that the participants referred to when describing fractions was not substantial, ranging from mentions from 33 participants for F4 to 48 participants for F2. Nonetheless, the picture is different when looking at how their descriptions were spread within the categories. All the subcategories in F2 and F3 were represented in the descriptions, although not consistently. For those categories, Comparing fractions in F3 was mentioned the least, whereas Symbolic Representations in F2 was mentioned the most. Conversely, in categories F1 and F4, several core interpretations and fraction notions were not included in the participants' conceptions. For example, only two of the seven interpretations in F1, that is, part-whole and quotient, were mentioned as fraction interpretations. Moreover, while some participants gave comparatively detailed descriptions and examples of the fraction aspects also covering most subcategories, some other participants provided very limited descriptions of the content in different subcategories.

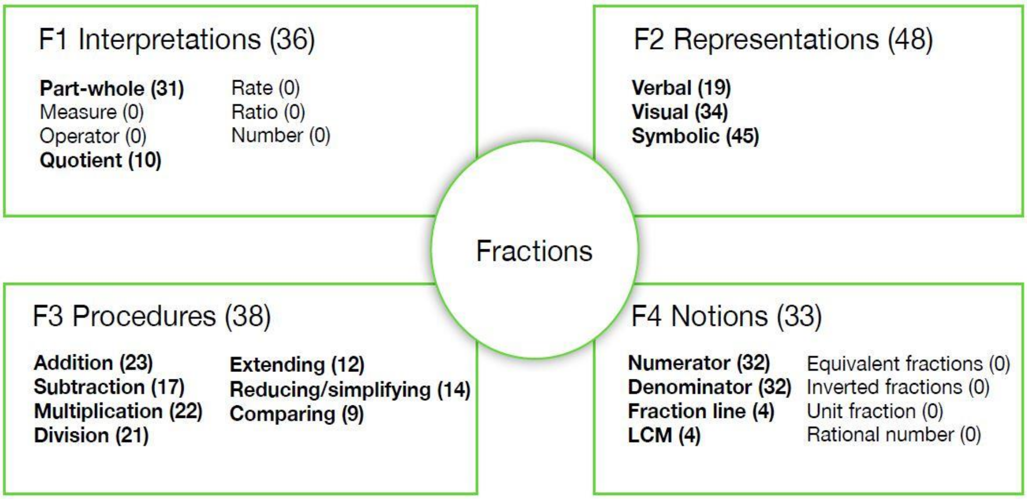


Figure 2. An illustration of the participating student teachers' conceptions of fractions. Numbers in parentheses represent the number of the participants referring to a category or a subcategory.

F1 Interpretations

The part-whole model of fractions dominated as a fraction interpretation in the participants' descriptions. More than half of the student teachers (*n* = 31) referred to this interpretation, describing, for example, that "a fraction is parts of something", "a part of the whole" or "different parts". The notion part-whole model was not used as such, but it was referred to by many participants using formulas like $\frac{\text{part}}{\text{the whole}}$ that also refer to the form of fractions. Ten participants also referred to the quotient interpretation, that is, a fraction as the result of division.

Altogether, 21 participants did not provide any connections to this F1 category of what fractions are. Moreover, the fundamental gap identified in F1 was that none of the participants provided an interpretation of fractions as a measure, operator, rate, ratio, or a separate number. Some identified gaps also concerned the participants' use of the part-whole model and the unit used with fractions. In the participants' descriptions, fractions were mainly interpreted as a quantity that was divided into some number of equal-sized parts of which some number of parts were then taken, for example, $\frac{2}{4}$ of a rectangle. The participants expressed that this quantity is "the whole", and it was related to the whole number 1. Since fractions were associated with the whole number 1, only this number or one figure was used as the unit in their examples and visual representations when referring to the part-whole interpretation.

F2 Representations

Most participants (48 of 57) connected different representations and forms to fractions. Mathematical symbolic representations, such as $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{6}$, and $\frac{1}{10}$, were most common in their descriptions. Some participants also used written forms like $\frac{\text{numerator}}{\text{denominator}}$ to present the fraction form without giving any examples of an actual fraction. Written verbal representations, such as a half, a third, a fourth, and one-fifth were also used, and a couple of participants also provided examples of how to pronounce a fraction in different ways. Fractions were mainly considered as unit fractions. Non-unit fractions that have a number other than 1 in the numerator, such as $\frac{3}{4}$ and $\frac{2}{5}$ were only provided with some pictorial representations and when illustrating fraction procedures. Moreover, examples of more complicated fractions like a fraction with a two-digit number in the numerator and the denominator were not presented, which can be interpreted as a gap in this category.

Another limitation identified in F2 was the participants' preference for constructing all the fractions using the form $\frac{a}{b}$; the form a/b was not represented in their descriptions. In Swedish school mathematics, both forms denote fractions and division. The whole number 1 was presented as different fractions, such as $\frac{1}{1}$ and $\frac{2}{2}$. Other whole numbers, however, were not converted to fractions, showing, for example, $4 = \frac{4}{1} = \frac{8}{2}$. One participant described the connection between fraction forms and whole numbers stating incorrectly: " $\frac{1}{1}$ or $\frac{3}{3}$ is the whole, that is, 1 or 3." These findings indicate limitations and gaps in the participants' conceptions concerning the connection between fractions and natural numbers.

Fractions were also connected to percentage and decimal forms, which were presented with written and mathematical symbolic representations (see Figure 3). Only six participants, however, connected fractions to both a percentage form and a decimal form. The percentage form was described as a support when calculating fraction tasks, and one participant described the relation between fractions and decimals stating that fractions "can represent something more precise than what decimal numbers can do, for example, $\frac{1}{3} \approx 0.33$." Some participants also mentioned the mixed number form, but there were few descriptions of the connection between fractions and mixed numbers. Mixed numbers were mainly provided with examples of adding and multiplying fractions, and not any pictorial examples of mixed numbers were given.

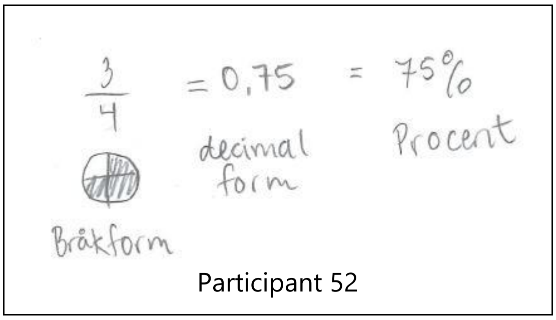


Figure 3. A fraction presented with different representations and forms ("Bråkform" is the Swedish expression for fraction form).

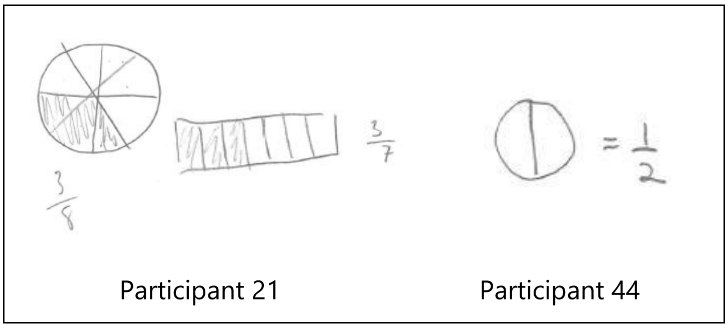
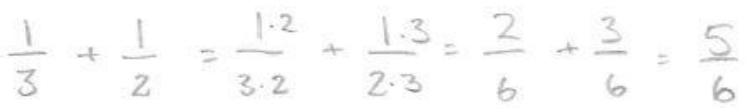


Figure 4. Pictorial representations with non-equal size and non-coloured parts.

Altogether, 34 participants provided drawings as visual representations for fractions, and seven of them also mentioned manipulatives and some concrete materials, such as fraction bars and circles, cubes, pizzas, and cake slices. The drawings that illustrated fractions were mainly separate coloured pie diagrams and rectangles. The participants did not provide sets of drawn objects like a group of four apples. In their written descriptions, the participants highlighted same-sized parts for fractions, but many of their drawings did not include equal-sized parts (see Figure 4). Moreover, in some participants' illustrations, the drawings and the fractions connected to them did not seem to be equivalent as seen in Figure 4. The discrepencies identified in the participants' pictorial representations might cause confusion when teaching elementary school students.

F3 Procedures

In the F3 category, all four operations, that is, addition, subtraction, multiplication, and division, were connected to fractions. Addition was most often mentioned in the participants' descriptions, and it was typically used in the examples of fraction calculations (see Figure 5).



Participant 50

Figure 5. An example of extending fractions in addition.

Some participants referred to the extending and reducing procedures mentioning the need to have the same denominators for fraction addition and subtraction, which was also described as the meaning of the extending process. Few participants, however, provided the procedures for extending or reducing fractions, like in the example in Figure 5. Some participants also referred to reducing in the sense of getting the simplest form for a fraction, writing, for example, $\frac{2}{4} = \frac{1}{2}$. The relation between different fractions was often described as follows: $\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{4}{8} = \frac{5}{10}$. Comparing fractions, however, was usually connected to the extending and reducing procedures that do not change the value of the fractions. Thus, the participants did not provide any procedures or descriptions of how to compare the sizes of different fractions.

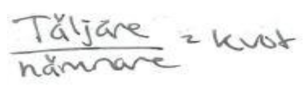
Concerning the F3 category in general, the participants mentioned different procedures for fractions rather than presented how to actually work through the procedures with fractions. Some participants even described their uncertainty using different fraction procedures, especially multiplication and division were challenging for many student teachers as can be seen in the mistakes in Figure 6. Moreover, 19 participants did not refer to any fraction procedures defined for this category.

Participant 14

Figure 6. Gaps in multiplication and division with fractions.

F4 Notions

The F4 category, how to characterise fractions with different notions, was the least represented in the participating student teachers' conceptions of fractions. Altogether, 33 of the 57 participants referred to F4. Their descriptions did not cover the notions of a rational number, unit fraction, or equivalent and inverted fractions, which were included in the framework for the category. Thirty-two participants connected the core notions numerator (”täljare”) and denominator (”nämnare”) to fractions, and some of them also used the Swedish term “kvot” that refers to the result of a division (see Figure 7). Moreover, when interpreting fractions as quotients of two integers, two participants also used the notions dividend and divisor.



Participant 40

Figure 7. An example of notions connected to the fraction form.

Fraction line was used in the participants' descriptions when illustrating the fraction form (see e.g., Figure 7), but the notion itself was only provided by four participants. Similarly, four participants connected the notion of least common multiple (LCM) to fractions when adding fractions with different denominators. In general, few correct mathematical notions were provided in connection to fractions. For example, few participants used the notions of a mixed number and a simplified fraction even though these fraction forms were represented in their examples of different procedures. Some participants also used informal expressions that refer to some mental images, which are given to fraction notions and often used in Swedish school mathematics, such as the Swedish words “tak” (roof) and ”nedre våning” (lower floor) that illustrate the places for the numerator and denominator in the fraction construction $\frac{a}{b}$.

Summary of the Results

As a summary, Figure 8 presents a graphical illustration of how individual participant fraction descriptions were spread both between and within the four categories F1–F4. The participants' references to each subcategory are presented in Figure 8 with different shades of a particular colour per category, and the gaps in their conceptions of fractions are illustrated as white squares. Thus, it can be seen in the white columns that the categories focusing on interpretations (F1) and notions (F4) both contain a lot of gaps whereas the categories of representations (F2) and procedures (F3) are better represented in the participants' conceptions. Note that Respondents 32, 36, 42, and 46 did not answer the questionnaire section concerned in this study and are hence not represented in Figure 8.

Moreover, when looking at Figure 8 by its 57 rows (where each row represents a participating student teacher), more observations can be made concerning individual participants' conceptions of fractions. For example, only 14 participants provided descriptions representing content relating to all the fraction aspects, whereas 21 of the 57 participants referred to three categories. Seven participants only referred to one category, like Participant 20 (procedures) and Participant 52 (representations). Further, the horizontal gaps also illustrate differences between the participants in their conceptions of fractions within the subcategories. For example, in F3, only Participant 34 and Particpant 61 referred to all the seven subcategories (i.e., addition, subtraction, multiplication, division, extending, reducing, and comparing fractions), while 11 participants only desribed one fraction procedure in the same category. Figure 8 also reveals that a student teacher may demonstrate a comprehensive conception related to one fraction category while the same participant's descriptions in other categories may include severe gaps (cf., Participant 61 in F3 and other categories).

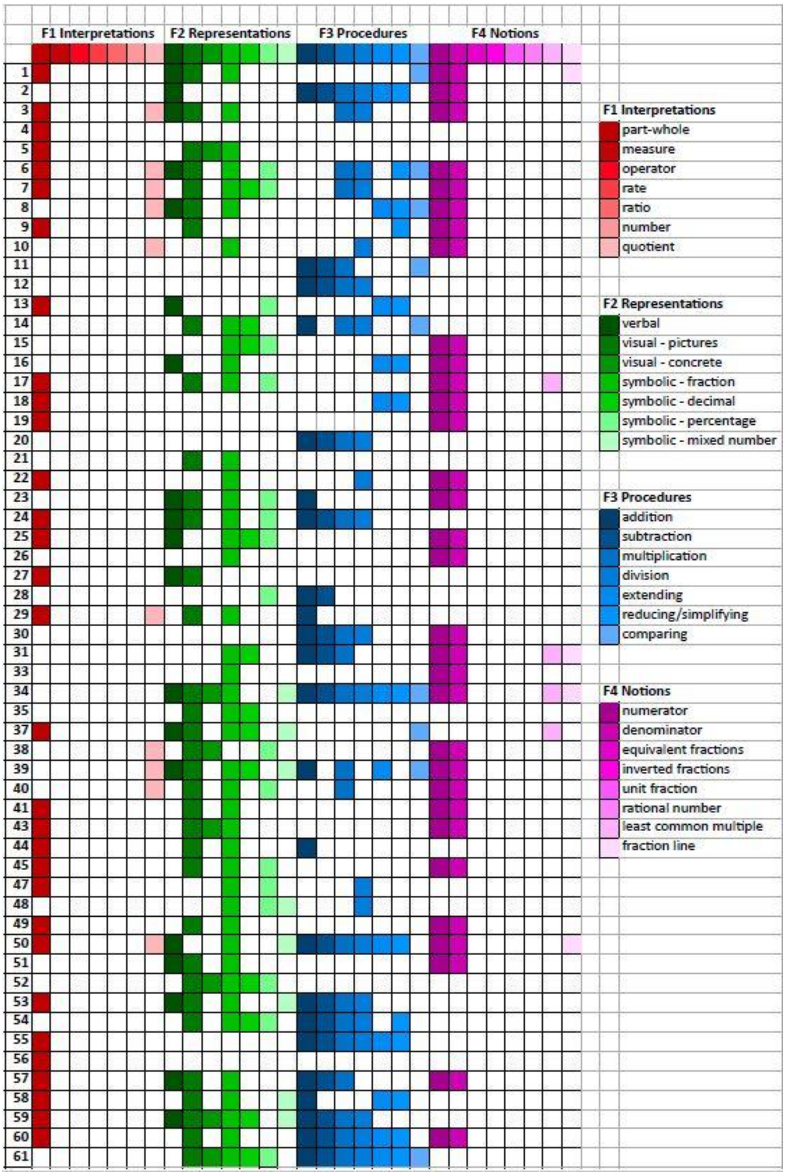


Figure 8. The ID codes for the 57 participants and each participant's fraction descriptions categorised according to the subcategories in F1–F4.

Discussion and Implications

This study investigated elementary school student teachers' conceptions of fractions by analysing their written fraction descriptions. The overall aim was to investigate how student teachers demonstrate their conceptual knowledge of fractions when asked to relate the concept of fractions to other concepts and mathematical constructions. While previous studies (e.g., Charalambous & Pitta-Pantazi, 2007) have mainly focused on fraction interpretations and operations, this study broadens the view of the topic by considering four aspects of fractions with a new analytical framework. Using the devised framework in an interpretive manner, we focused on the student teachers' conceptual knowledge by giving them ample time to describe concepts, terms, and notions associated with fractions. We were also able to identify gaps in their conceptions of fractions. Next, we will discuss the main findings related to the two reseach questions. Then, we will present some implications of the results.

Student Teachers' References to Different Aspects of Fractions

In their descriptions, the participants referred to all four aspects of fractions. They mainly provided conceptions that demonstrated different representations and procedures for fractions, and their descriptions involved all the subcategories included in the analytical framework for these categories. Instead, a deeper conceptual knowledge of what fractions are (interpretations) and what characterises fractions (notions) was not identified to a similar extent in their conceptions. It is hence reasonable to assume that the participating student teachers' conceptions of fractions consist of more procedural than conceptual aspects of fractions, which is also in line with previous research (e.g., Chinnappan & Forrester, 2014).

The participants showed awareness of mathematical symbolic representations and different visual representations of fractions, that is, pictorial and concrete models to illustrate fractions. It was also common for them to mention the core notions of a numerator and denominator and to refer to the part-whole interpretation of fractions. The findings also reveal substantial differences between the student teachers in their conceptions of fractions. Even though attending the same teacher education, some participants' descriptions indicated more comprehensive conceptions than other student teachers' few references and short answers, which is an interesting finding that pinpoints some challenges teacher education may have when developing student teachers with different backgrounds and prior knowledge in mathematics.

Gaps in Student Teachers' Conceptions of Fractions

In general, the participating student teachers' conceptions of fractions included several gaps. Their limited references within each of the F1–F4 subcategories constitute the main gaps in knowledge and understanding. Even though all the subcategories in F2 and F3 were referred to by some participants, no subcategory within F1–F4 was mentioned by all participants. The gaps were identified by the absence of mentions of several core interpretations and notions.

The overrepresentation of the part-whole interpretation in F1, and associated underrepresentation of other interpretations, can be seen as a gap in the participants' conceptions of fractions. Previous research (e.g., Behr et al., 1983; Charalambous & Pitta-Pantazi, 2007; Olanoff et al., 2014) has also discussed the dominance of the part-whole model, which can hinder the learning of fractions (Lovin et al., 2018; Rønning, 2013). For example, Thomson and Saldanha (2003) stated that having access to only a part-whole conceptualisation of fractions makes multiplicative reasoning with fractions intuitively impossible. Another gap related to interpretations concerns the difficulty in identifying the unit for a fraction, which has also been shown in previous research (e.g., Lamon, 2020). In this study, student teachers only used the whole number 1 or one object as the unit. Thus, when they only drew part-whole figures, the conceptual idea of improper fractions, such as $\frac{7}{4}$, is hard to realise since the fraction seems to consist of more parts than what is included in the unit. Taking some number other than 1 as the unit and dividing the set of objects (e.g., eight circles) into same size groups was also missing from the participants' descriptions. Moreover, the participants associated the meaning of the denominator with the number of parts that the whole unit was divided into. This does not make sense in the case of simplified fractions where the original unit (e.g., a set of objects) is reduced. As Lamon (2020) stated, students should quickly learn "that the unit may be something different in every new context and that the first question they [children] should always ask themselves is, 'What is the unit?'" (pp. 102–103).

Some participants also expressed their uncertainty concerning different procedures for fractions. They had difficulties presenting how to work through fraction procedures, especially the multiplication and division operation. This result aligns with previous research (e.g., Son & Lee, 2016; Tirosh, 2000). Moreover, when making mistakes with the procedures, the student teachers did not notice the unreasonable answers they provided with their examples. These findings also align with previous research findings (e.g., Newton, 2008; Tossavainen, 2022; Young & Zientek, 2011). Further, the participants provided pictorial illustrations that were not divided into equal-sized parts, which might cause misconceptions later in their teaching of fractions. Similarly, their limited use of proper notions with fractions might be a hindrance to effective teaching and learning of fractions. As Kaiser et al. (2017) have shown, teachers' mathematical knowledge does not change much during the first years in the profession. Thus, it is a challenge to teacher education to help student teachers to overcome the gaps and limitations in their fraction knowledge and to gain the knowledge needed.

Implications

The researchers involved in this study did not intend to evaluate the learning processes that have resulted in the identified conceptions and gaps or to compare the participating student teachers. However, the substantial differences found between the individual participants raise some concerns regarding the way teacher education addresses student teachers' individual differences. An interesting direction for future research would therefore be to develop and test specific interventions targeting these differences and individual needs that student teachers bring with them to their teacher education studies. Stolmann et al. (2014) provided an insight into one possibility, showing that by focusing on mathematical topics that student teachers understand mainly procedurally, for example, fraction division, is a way to change their conceptual understanding. Moreover, when enhancing student teachers' conceptual knowledge of fractions, the importance of multiple representations in the teaching of the challenging contents of fractions needs to be realised (Stolmann et al., 2014). For example, teacher education should provide student teachers with opportunities to solve fraction tasks in different contexts with multiple visual representations, explore connections between fraction representations and make transitions from one representation to another, analyse good and poor-quality representations, as well as to interpret and respond to elementary students' thinking in different contexts to overcome misconceptions concerning fractions (Son & Lee, 2016; Stevens et al., 2020; Thurtell et al., 2019). We also think that this should be done constructing a coherent understanding so that the meaning of all contexts used and representations together, build a coherent network rather than isolated islands of understanding (Thompson, 2013).

Further, an interesting question is how teacher education can ensure that student teachers are able to later transfer the enhanced conceptual knowledge of fractions to their teaching practices. Various scholars (e.g., Lamberg & Wiest, 2015; Van Steenbrugge et al., 2014) pointed out the need for sufficient time when working with fractions, which should also be addressed in teacher education when improving student teachers' conceptual knowledge. In line with the arguments by Juter (2022) concerning student teachers' knowledge of real numbers, highlighted in the current study is the need for student teachers to become aware of their mathematical knowledge and limitations in the fractional number domain.

Conclusion and Limitations

Sfard (1991) stated that conceptual knowledge develops through procedural knowledge. Taking the findings presented here into account, it can also be concluded that if student teachers do not have a comprehensive procedural fraction knowledge, they also have difficulties in the conceptual knowledge domain (Chinnappan & Forrester, 2014). This study also confirms the conclusion of Charalambous and Pitta-Pantazi (2007), namely that there is a need of emphasising other fraction interpretations than the part-whole model, and that the use of core notions and proper language with fractions should also be highlighted in teacher education (Stevens et al., 2020). In general, teacher education should focus more on the conceptual knowledge of fractions than on different algorithms to execute operations on fractions (Charalambous & Pitta-Pantazi, 2007; Copur-Gencturk, 2021). As Chinnappan and Forrester (2014) stated, student teachers' procedural-based fraction knowledge can be an obstacle to the development of specialised content knowledge and pedagogical content knowledge that are needed in the teaching of fractions. If our results are representative, there may be a need to revise the ways fractions are introduced in teacher education.

A limitation of the current study is that we only investigated one cohort of student teachers attending one Swedish university. Moreover, we do not know whether their answers in the paper-and-pencil questionnaire really represented their conceptions of fractions. The way our results coincide with previous research, however, might indicate that the reported findings of student teachers' conceptions of fractions can also be considered in other teacher education contexts. Thus, as a consequence of our results, we would encourage teacher educators to examine carefully their student teachers' fraction knowledge before assuming they have the knowledge needed to teach fractions in their future professions as mathematics teachers. The F1–F4 framework developed for this study may be a useful tool for the assessment of that fraction knowedge.

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Ethics Declarations

Ethical approval

The research reported in this paper complied with the *Guidance On Ethical Review of Research on Humans* published by the Swedish Ethical Review Authority. Informed consent was given by all participants for their data to be published.

Competing interests

The authors declare there are no competing interests.

References

Agathangelou, S. A., & Charalambous, C. Y. (2021). Is content knowledge pre-requisite of pedagogical content knowledge? An empirical investigation. *Journal of Mathematics Teacher Education, 24*(5), 431–458. https://doi.org/10.1007/s10857-020-09466-0

Ahl, L. M., & Helenius, O. (2021a). A framework for analyzing progress in concept knowledge in mathematics textbooks. In G. A. Nortvedt, N. A. Buchholtz, J. Fauskanger, F. Hreinsdóttir, M. Hähkiöniemi, B. E. Jessen, J. Kurvits, Y. Liljekvist, M. Misfeldt, M. Naalsund, H. K. Nilsen, G. Pálsdóttir, P. Portaankorva-Koivisto, J. Radišić, & A. Wernberg (Eds.), *Bringing Nordic mathematics education into the future. Proceedings of NORMA 20: The Ninth Nordic Conference on Mathematics Education*, Oslo (pp. 1–8). SMDF.

Ahl, L. M., & Helenius, O. (2021b). Polysemy and the role of representations for progress in concept knowledge. In Y. Liljekvist & J. Häggström (Eds.), *Sustainable mathematics education in a digitalized world. Proceedings of MADIF 12* (pp. 101–110). SMDF.

Ainsworth, S. (2006). DeFT: A conceptual framework for considering learning with multiple representations. *Learning and Instruction, 16*(3), 183–198. https://doi.org/10.1016/j.learninstruc.2006.03.001

Anghileri, J. (2006). Scaffolding practices that enhance mathematics learning. *Journal of Mathematics Teacher Education, 9*(1), 33–52. https://doi.org/10.1007/s10857-006-9005-9

Ball, D. L. (1993). Halves, pieces, and twoths: Constructing representational contexts in teaching fractions. In T. P. Carpenter, E. Fennema, & T. A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 157–196). Lawrence Erlbaum Associates.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*(5), 389–407. https://doi-org.proxy.lib.ltu.se/10.1177/0022487108324554

Bansilal, S., & Ubah, I. (2020). The use of cross multiplication and other mal-rules in fraction operations by pre-service teachers. *The Journal of Mathematical Behavior, 58*, Article 100781. https://doi.org/10.1016/j.jmathb.2020.100781

Behr, M., Lesh, R., Post, T., & Silver, E. (1983). Rational number concepts. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 91–125). Academic Press.

Behr, M. J., Wachsmuth, I., Post, T. R., & Lesh, R. (1984). Order and equivalence of rational numbers: A clinical teaching experiment. *Journal for Research in Mathematics Education, 15*(5), 323–341. https://doi.org/10.2307/748423

Bray, W. S. (2011). A collective case study of the influence of teachers' beliefs and knowledge on error-handling practices during class discussion of mathematics. *Journal for Research in Mathematics Education, 42*(1), 2–38. https://doi.org/10.5951/jresematheduc.42.1.0002

Charalambous, C. Y., & Pitta-Pantazi, D. (2007). Drawing on a theoretical model to study students' understandings of fractions. *Educational Studies in Mathematics, 64*(3), 293–316. https://doi.org/10.1007/s10649-006-9036-2

Chinnappan, M., & Forrester, T. (2014). Generating procedural and conceptual knowledge of fractions by pre-service teachers. *Mathematics Education Research Journal, 26*(4), 871–896. https://doi.org/10.1007/s13394-014-0131-x

Copur-Genturk, Y. (2021). Teachers' conceptual understanding of fraction operations: Results from a national sample of elementary school teachers. *Educational Studies in Mathematics, 107*(3), 525–545. https://doi.org/10.1007/s10649-021-10033-4

Cramer, K. A., Post, T. R., & delMas, R. C. (2002). Initial fraction learning by fourth- and fifth-grade students: A comparison of the effects of using commercial curricula with the effects of using the rational number project curriculum. *Journal for Research in Mathematics Education, 33*(2), 111–144. https://doi.org/10.2307/749646

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics, 61*, 103–131. https://doi.org/10.1007/s10649-006-0400-z

Ebbelind, A., Roos, H., & Nilsson, P. (2012). Learning fractions: Transformations between representations from a social semiotic perspective of multimodality. In G. H. Gunnarsdottir, F. Hreindottir, G. Palsdottir, M. Hannula, M. Hannula-Sormunen, E. Jablonka, U. T. V. Jankvist, A. Ryve, P. Valero & K. Waege (Eds.), *Proceedings of NORMA 11: The Sixth Nordic Conference on Mathematics Education* (pp. 217–226). University of Iceland Press.

Getenet, S., & Callingham, R. (2021). Teaching interrelated concepts of fraction for understanding and teacher's pedagogical content knowledge. *Mathematics Education Research Journal, 33*(2), 201–221. https://doi.org/10.1007/s13394-019-00275-0

Graeber, A. (1999). Forms of knowing mathematics: What preservice teachers should learn. *Educational Studies in Mathematics, 38*(1–3), 189–208. https://doi-org.proxy.lib.ltu.se/10.1023/A:1003624216201

Gray, E. M., & Tall, D. O. (1994). Duality, ambiguity, and flexibility: A "proceptual" view of simple arithmetic. *Journal for Research in Mathematics Education, 25*(2), 116–140. https://doi.org/10.2307/749505

Hallett, D., Nunes, T. & Bryant, P. (2010). Individual differences in conceptual and procedural knowledge when learning fractions. *Journal of Educational Psychology, 102*(2), 395–406. https://doi.org/10.1037/a0017486

Hiebert, J., & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: An introductory analysis. In J. Hiebert (Ed.) *Conceptual and procedural knowledge: The case of mathematics* (pp. 1–27). Lawrence Erlbaum.

Ibañez, E. D., & Pentang, J. T. (2021). Socio-constructivist learning and teacher education students' conceptual understanding and attitude toward fractions. *Indonesian Research Journal in Education, 5*(1), 23–44. https://doi.org/10.22437/irje.v5i1.12187

Jacobson, E., & Izsák, A. (2015). Knowledge and motivation as mediators in mathematics teaching practice: The case of drawn models for fraction arithmetic. *Journal of Mathematics Teacher Education, 18*(5), 467–488. https://doi-org.proxy.lib.ltu.se/10.1007/s10857-015-9320-0

Jakobsen, A., Ribeiro, C. M., & Mellone, M. (2014). Norwegian prospective teachers' MKT when interpreting pupils' productions on a fraction task. *Nordic Studies in Mathematics Education, 19*(3–4), 135–150.

Jóhannsdóttir, B., & Gísladóttir, B. (2014). Exploring the mathematical knowledge of prospective elementary teachers in Iceland using the MKT measures. *Nordic Studies in Mathematics Education, 19*(3–4), 21–40.

Joutsenlahti, J., & Perkkilä, P. (2021). Murtoluku vai suhde? [A fraction or a ratio?]. *FMSERA Journal, 4*(1), 61–74. https://journal.fi/fmsera/article/view/95634

Juter, K. (2022). Pre-service teachers' general and specific arguments in real number contexts. *Mathematics Teacher Education and Development*, *24*(1), 1–22. https://mted.merga.net.au/index.php/mted/article/view/699

Kaiser, G., Blömeke, S., König, J., Busse, A., Döhrmann, M., & Hoth, J. (2017). Professional competencies of (prospective) mathematics teachers: Cognitive versus situated approaches. *Educational Studies in Mathematics, 94*(2), 161–184. https://doi.org/10.2307/45185357

Kieren, T. E. (1976). On the mathematical, cognitive, and instructional foundations of rational numbers. In R. Lesh (Ed.), *Number and measurement: Papers from a research workshop* (pp. 101–144). Columbus.

Kieren, T. E. (1993). Rational and fractional numbers: From quotient fields to recursive understanding. In T. P. Carpenter, E. Fennema & T. A. Romberg (Eds.) *Rational numbers: An integration of research* (pp. 49–84). Lawrence Erlbaum Associates.

Lamberg, T., & Wiest, L. R. (2015). Dividing fractions using an area model: A look at in-service teachers' understanding. *Mathematics Teacher Education and Development, 17*(1), 30–43. https://mted.merga.net.au/index.php/mted/article/view/176

Lamon, S. J. (2007). Rational numbers and proportional reasoning: Toward a theoretical framework for research. In F. K. Lester, Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 629–667). Information Age Publishing.

Lamon, S. J. (2020). *Teaching fractions and ratios for understanding: Essential content knowledge and instructional strategies for teachers* (4th ed.). Routledge.

Lee, J.-E., & Lee, M. Y. (2023). How elementary prospective teachers use three fraction models: their perceptions and difficulties. *Journal of Mathematics Teacher Education, 26*(4), 455–480. https://doi.org/10.1007/s10857-022-09537-4

Lin, C.-Y., Becker, J., Byun, M.-R., Yang, D.-C., & Huang, T.-W. (2013). Preservice teachers' conceptual and procedural knowledge of fraction operations: A comparative study of the United States and Taiwan. *School Science and Mathematics, 113*(1), 41–51. https://doi.org/10.1111/j.1949-8594.2012.00173.x

López-Martín, M.d.M., Aguayo-Arriagada, C. G., & García López, M.d.M. (2022). Preservice elementary teachers' mathematical knowledge on fractions as operator in word problems. *Mathematics, 10*(3), 423. https://doi.org/10.3390/math10030423

Lortie-Forgues, H., Tian, J., & Siegler, R. S. (2015). Why is learning fraction and decimal arithmetic so difficult? *Developmental Review, 38*, 201–221. https://doi.org/10.1016/j.dr.2015.07.008

Lovin, L. H., Stevens, A. L., Siegfried, J., Wilkins, J. L. M., & Norton, A. (2018). Pre-K–8 prospective teachers' understanding of fractions: An extension of fractions schemes and operations research. *Journal of Mathematics Teacher Education, 21*(3), 207–235. https://doi.org/10.1007/s10857-016-9357-8

Löwing, M. (2016). *Diamant—diagnoser i matematik. Ett kartläggningsmaterial baserat på didaktisk ämnesanalys* [*Diamant—Diagnoses in mathematics. A screening material based on a didactical subject analysis*]. Acta Universitatis Gothoburgensis. http://hdl.handle.net/2077/47607

Marchionda, H. (2006). *Preservice teachers' procedural and conceptual understanding of fractions and the effects of inquiry-based learning on this understanding* [Doctoral dissertation, Clemson University]. https://tigerprints.clemson.edu/all_dissertations/37

Marshall, S. P. (1993). Assessment of rational number understanding: A schema-based approach. In T. P. Carpenter, E. Fennema, & T. A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 261–288). Lawrence Erlbaum Associates.

Muir, T., & Livy, S. (2012). What do they know? A comparison of pre-service teachers' and in-service teachers' decimal mathematical content knowledge. *International Journal for Mathematics Teaching and Learning, 2012,* December 5th, 1–15. http://www.cimt.org.uk/journal/muir2.pdf

Newton, K. J. (2008). An extensive analysis of preservice elementary teachers' knowledge of fractions. *American Educational Research Journal, 45*(4), 1080–1110. https://doi.org/10.3102/0002831208320851

Ohlsson, S. (1988). Mathematical meaning and applicational meaning in the semantics of fractions and related concepts. In J. Hiebert & M. Behr (Eds.) *Number concepts and operations in the middle grades: Volume 2*. (pp. 53–92). Lawrence Erlbaum Associates.

Olanoff, D., Lo, J.-J., & Tobias, J. (2014). Mathematical content knowledge for teaching elementary mathematics: A focus on fractions. *The Mathematics Enthusiast, 11*(2), 267–310. https://doi.org/10.54870/1551-3440.1304

Pantziara, M., & Philippou, G. (2012). Levels of students' "conception" of fractions. *Educational Studies in Mathematics, 79*(1), 61–83. https://doi.org/10.1007/s10649-011-9338-x

Rønning, F. (2013). Making sense of fractions given with different semiotic representations. In B. Ubuz, Ç. Haser, & M. A. Mariotti (Eds.), *Proceedings of the Eight Congress of the European Society for Research in Mathematics Education* (pp. 1586–1595). Middle East Technical University, Ankara.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 22*(1), 1–36. https://doi.org/10.1007/BF00302715

Siegler, R. S., & Braithwaite, D. W. (2017). Numerical development. *Annual Review of Psychology, 68*(1), 187–213. https://doi.org/10.1146/annurev-psych-010416-044101

Skolverket. [Swedish National Agency for Education]. (2022a). *Läroplan för grundskolan, förskoleklassen och fritidshemmet—Lgr22* [Curriculum for the compulsory school, preschool class and school-age educare]. Skolverket.

Skolverket. [Swedish National Agency for Education]. (2022b). *Kommentarmaterial till kursplanen i matematik. Grundskolan* [*Commentary material for the syllabus in mathematics.: Primary School*].

Son, J.-W., & Lee, J.-E. (2016). Pre-service teachers' understanding of fraction multiplication, representational knowledge, and computational skills. *Mathematics Teacher Education and Development, 18*(2), 5–28. https://mted.merga.net.au/index.php/mted/article/view/230

Stohlmann, M., Cramer, K., Moore, T., & Maiorca, C. (2014). Changing pre-service elementary teachers' beliefs about mathematical knowledge. *Mathematics Teacher Education and Development*, *16*(2), 4–24. https://mted.merga.net.au/index.php/mted/article/view/246

Stevens, A. L., Wilkins, J. L. M., Lovin, L. H., Siegfried, J., Norton, A., & Busi, R. (2020). Promoting sophisticated fraction constructs through instructional changes in a mathematics course for PreK–8 prospective teachers. *Journal of Mathematics Teacher Education, 23*(2), 153–181. https://doi.org/10.1007/s10857-018-9415-5

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*(2), 151–169. https://doi.org/10.1007/BF00305619

Tchoshanov, M. A. (2011). Relationship between teacher knowledge of concepts and connections, teaching practice, and student achievement in middle grades mathematics. *Educational Studies in Mathematics, 76*(2), 141–164. https://doi-org.proxy.lib.ltu.se/10.1007/s10649-010-9269-y

Thurtell, E., Forrester, T., & Chinnappan, M. (2019). Building conceptual knowledge of fraction operations among pre-service teachers: Effect of a representation-based teaching approach within a teacher education program. *Mathematics Teacher Education and Development, 21*(1), 100–124. https://mted.merga.net.au/index.php/mted/article/view/567

Tirosh, D. (2000). Enhancing prospective teachers' knowledge of children's conceptions: The case of division of fractions. *Journal for Research in Mathematics Education*, *31*(1), 5–25. https://doi.org/10.2307/749817

Thompson, P. W. (2013). In the absence of meaning …. In K. R. Leatham (Ed.), *Vital directions for mathematics education research* (pp. 57–93). Springer. https://doi.org/10.1007/978-1-4614-6977-3_4

Thompson, P. W., & Saldanha, L. A. (2003). Fractions and multiplicative reasoning. In J. Kilpatrick, G. W. Martin & D. Schifter (Eds.), *Research companion to the principles and standards for school mathematics* (pp. 95–113). National Council of Teachers of Mathematics.

Tobias, J. M. (2013). Prospective elementary teachers' development of fraction language for defining the whole. *Journal of Mathematics Teacher Education, 16*(2), 85–103. https://doi.org/10.1007/s10857-012-9212-5

Toluk-Uçar, Z. (2009). Developing pre-service teachers understanding of fractions through problem posing. *Teaching and Teacher Education, 25*(1), 166–175. https://doi.org/10.1016/j.tate.2008.08.003

Tossavainen, A. (2022). Student teachers' common content knowledge for solving routine fraction tasks. *LUMAT: International Journal on Math, Science and Technology Education, 10*(2), 256–280. https://doi.org/10.31129/LUMAT.10.2.1656

Van Steenbrugge, H., Lesage, E., Valcke, M., & Desoete, A. (2014). Preservice elementary school teachers' knowledge of fractions: A mirror of students' knowledge? *Journal of Curriculum Studies, 46*(1), 138–161. https://doi.org/10.1080/00220272.2013.839003

Vergnaud, G. (2009). The theory of conceptual fields. *Human Development, 52*(2), 83–94. https://www.jstor.org/stable/26764894

Young, E., & Zientek, L. R. (2011). Fraction operations: An examination of prospective teachers' errors, confidence, and bias. *Investigations in Mathematics Learning, 4*(1), 1–23. https://doi.org/10.1080/24727466.2011.11790307

Young-Loveridge, J., Bicknell, B., & Mills, J. (2012). The mathematical content knowledge and attitudes of New Zealand pre-service primary teachers. *Mathematics Teacher Education and Development, 14*(2), 28–49. https://mted.merga.net.au/index.php/mted/article/view/152